5. The Derivative; Differentiation laws (19/9/2019)

(no worksheets due to bad copier)

Goals.
(1) The derivative as a function
(2) Power laws and polynomials
(3) The product and quotient rules
(4) office hours 11:00-12:00

Last Time.

(1) Gluing of functions (to make f cts at x_0 need to make $\lim_{x \to x_0^-} f(x) = f(x_0) = \lim_{x \to x_0^+} f(x)$)

(2) IVT: to show f has a zero on $[a,b]$:
(*) check f cts there
(*) show f positive and negative there
(*) Invoke IVT.
(*) Play endgame:

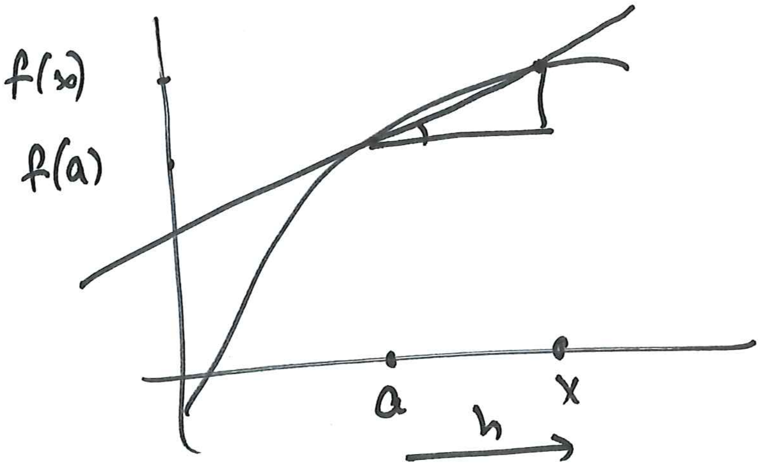
want solution to $x+1 = \sin x$
- let $f(x) = x+1-\sin x$
- choose a, b
⋮
- get x_0 s.t. $f(x_0) = 0$
$\Downarrow$
$x_0 + 1 = \sin x_0$

Difficulty: often endpoints a,b not given.
- choice arbitrary, forces us to deal with inequalities.

tips: (1) practice (2) goal: make true statements

3) <u>Derivative</u>: $f'(a) \overset{\text{def}}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Change of variable: $x - a = h$, $x = a + h$

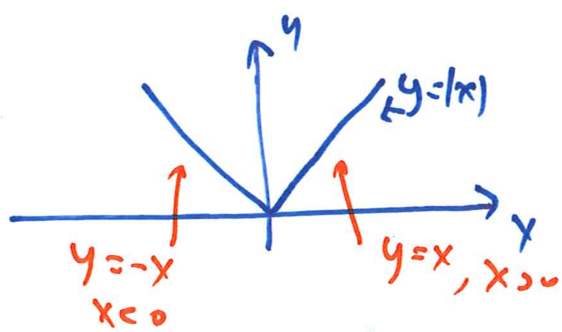


<u>Example</u>: Interpret the limit $\lim_{h \to 0} \frac{\cos(5+h) - \cos(5)}{h}$ as a derivative.

<u>Solution</u>: let $f(x) = \cos x$ then $f'(5) \overset{\text{by defn}}{=} \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{\cos(5+h) - \cos 5}{h}$

Suppose $f(x)$ is defined on (a,b), differentiable at all points of (a,b). We then call the function $x \mapsto f'(x)$ the <u>derivative of f</u>

<u>Example</u>: Consider $f(x) = |x|$

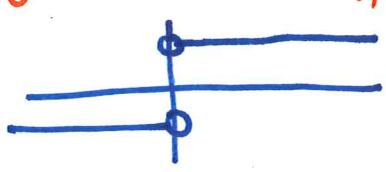


for $x > 0$, $f(x) = x$, has slope 1

for $x < 0$, $f(x) = -x$, has slope -1

At $x = 0$, f not diff: $\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 1$, $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -1$

so $f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undef} & x = 0 \end{cases}$



Examples: $f(x) = c$, $\displaystyle\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0$

write general def'n — plug in formula for f — evaluate limit

similarly get

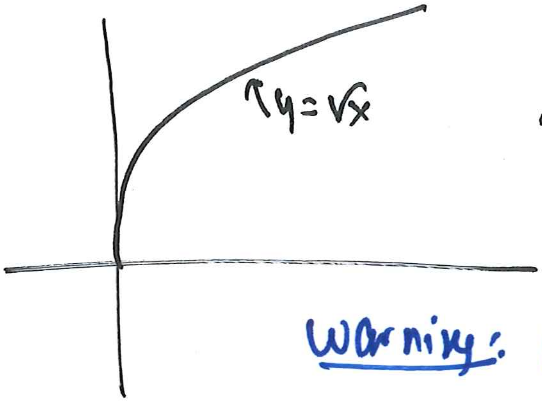
if $f(x) = x$, $f'(x) = 1$

$f(x) = x^2$, $f'(x) = 2x$

$f(x) = x^3$, $f'(x) = 3x^2$

If $f(x) = \sqrt{x}$, $\displaystyle\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$

$\displaystyle = \lim_{h\to 0} \frac{x+h-x}{h} \cdot \frac{1}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$



graph

warning: f' might have smaller domain

Fact: For any $r \in \mathbb{R}$, $\frac{d}{dx}(x^r) = r x^{r-1}$
(whenever x^r defined)
x^{r-1}

Fact: $\frac{d}{dx}(e^x) = e^x$

Fact: if f, g function, a, b constants $(af+bg)' = af' + bg'$

Math 100 – WORKSHEET 5
THE DERIVATIVE

1. Linear combinations; power laws

(1) If f, g are functions and a, b are numbers then $(af+bg)' = af' + bg'$

(2) $\frac{\mathrm{d}}{\mathrm{d}x}(x^r) = rx^{r-1}$ (3) $\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$.

(1)

(a) Differentiate $f(x) = \frac{5x^3-2x+1}{\sqrt{x}}$.

Note: $f(x) = 5\frac{x^3}{\sqrt{x}} - 2\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} = 5x^{5/2} - 2x^{1/2} + x^{-\frac{1}{2}}$

so $f'(x) = 5\cdot\frac{5}{2}x^{3/2} - 2\cdot\frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)x^{-3/2}$

$= \boxed{\frac{25}{2}x^{3/2} - x^{-\frac{1}{2}} - \frac{1}{2}x^{-3/2}} = \frac{25x^3 - 2x^2 - 1}{2x^{3/2}}$

if you want

(b) Let $g(x) = Ax^{5/2} + x^2$. Suppose that $g'(4) = 0$. What is A?

Plans
(1) diff g
(2) plug in $x=4$, get $g'(4)=0$ as an equation for A
(3) solve for A

here $g'(x) = \frac{5}{2}Ax^{3/2} + 2x$

if $g'(4)=0$ we have

$\frac{5}{2}\cdot 4^{3/2}\cdot A + 2\cdot 4 = 0$

i.e. $20A + 8 = 0$ so $\boxed{A = -\frac{2}{5}}$

Date: 19/9/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

New idea: If $f'(x)$ is a function, it may have a derivative.

Notation: f'' or $f^{(2)}$ or $\frac{d^2f}{dx^2}$

f''' or $f^{(3)}$ or $\frac{d^3f}{dx^3}$

Example: find f'' if

(1) $f(x) = e^x$

(2) $f(x) = \sqrt{x} + 5e^x$

$f'(x) = \frac{1}{2\sqrt{x}} + 5e^x$ so $f''(x) = -\frac{1}{4}x^{-3/2} + 5e^x$

(2) Find the *second* derivative of

(a) $5e^x$

(b) $\sqrt{x} + 5e^x$

(3) The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

Plan

(1) "names" say point of tangency (a, a^3+2a)

(2) line tangent to curve means: slope of line = slope of curve

so $\left.\frac{dy}{dx}\right|_{x=a} = 5$ is an equation for a

(3) at a, line meets curve: $a^3 + 2a = 5a + B$

that is an equation for B

slope of $y = x^3+2x$ at a

is $\left.\frac{dy}{dx}\right|_{x=a} = 3a^2 + 2$

So if slope is 5 we have

$3a^2 + 2 = 5$, ie. $a^2 = 1$,

i.e. $a = 1$ or $a = -1$

So we either have

$1^3 + 2 \cdot 1 = 5 + B$, ie $B = -2$

or

$(-1)^3 + 2(-1) = -5 + B$, & $B = 2$

Endgame: line $y = 5x - 2$ tangent at $(1,3)$

line $y = 5x + 2$ " " $(-1,-3)$