8. Inverse functions (1/10/2019)

Goals.

(1) Inverse functions
(2) Derivatives of inverse functions

Last Time.

Chain rule: if $h(x) = f(g(x))$ then $h'(x) = f'(g(x)) \cdot g'(x)$

$$\Leftrightarrow \frac{dh}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

Exercise: Suppose $f(g(x)) = x$. Find $g'(x)$ (in terms of f')

By chain rule, $f'(g(x)) \cdot g'(x) = 1$ so

$$\boxed{g'(x) = \frac{1}{f'(g(x))}}$$

("inverse function rule")

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worksheet sec 1

Math 100 – WORKSHEET 8
INVERSE FUNCTIONS

1. More on the chain rule

(1) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Solution: by chain rule $3x^2 = f'(g(x)) \cdot g'(x)$

At $x=4$, this reads: $f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$ so

$$g'(4) = \frac{48}{5}$$

2. Inverse functions

(1) Find the function inverse to $y = x^7 + 3$.

If $y = x^7 + 3$, then $x^7 = y - 3$, so $x = \sqrt[7]{y-3}$.

equiv., the inverse function is $y = \sqrt[7]{x-3}$

(2) Does $y = x^2$ have an inverse?

On domain $(-\infty, +\infty)$ no: multiple solutions to $y = x^2$ $(x = \pm\sqrt{y})$

On domain $[0, \infty)$ yes: only positive solution to $y = x^2$ is $\sqrt{y}$.

Date: 1/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Inverse function

Want to "undo f".

(1) operationally: to undo $x \mapsto x+5$ do $y \mapsto y-5$

to undo $x \mapsto x \cdot 5$ do $y \mapsto y/5$

to undo $x \mapsto x^5$ do $y \mapsto y^{1/5}$.

⋮

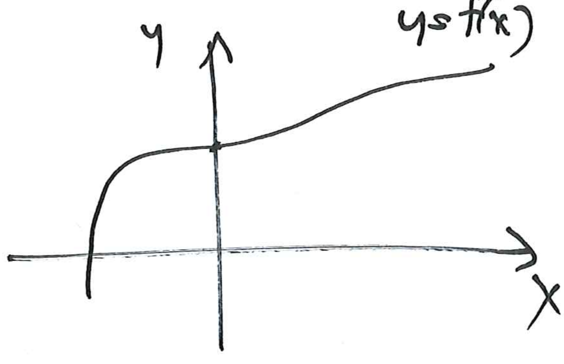
(2) solving equations: if $y = x+5$, what is x?

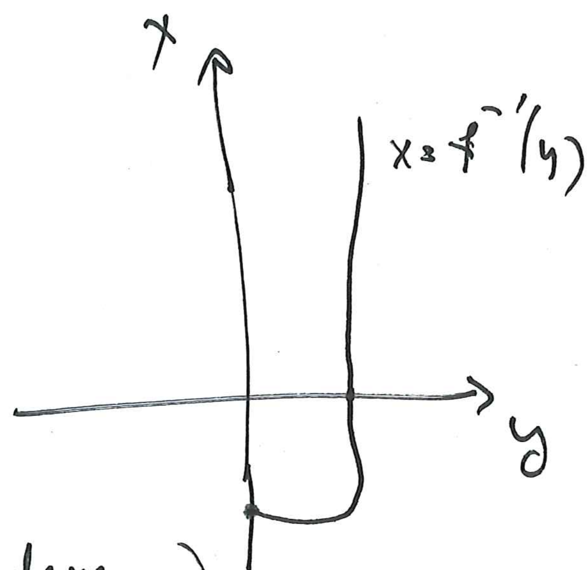
if $y = 5x$, what is x?

if $y = x^5$, what is x?

if $y = ax^2 + bx$, what is x?

(3) Geometrically:





("flip axes", or "rotate by 90 degrees & reflect in x axis")

Always the question is: given y value, what x value gave rise to it?

Notation: sometime use f^{-1} for the inverse function:

if $f(x) = y$ then $x = f^{-1}(y)$

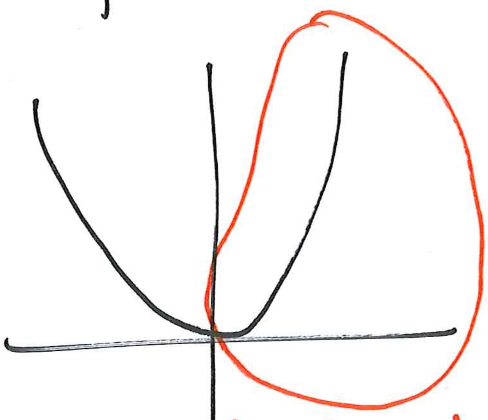
Example: Say $f(x) = 2 + x + \sin x$. Find $f^{-1}(2+\pi)$

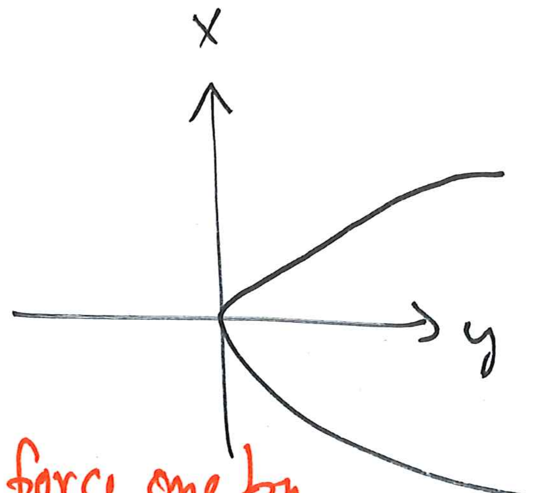
Try $x = 0$, here $f(0) = 2 + 0 + 0 = 2$

but $f(\pi) = 2 + \pi + 0 = 2 + \pi$ (← "the image of π is $2+\pi$")

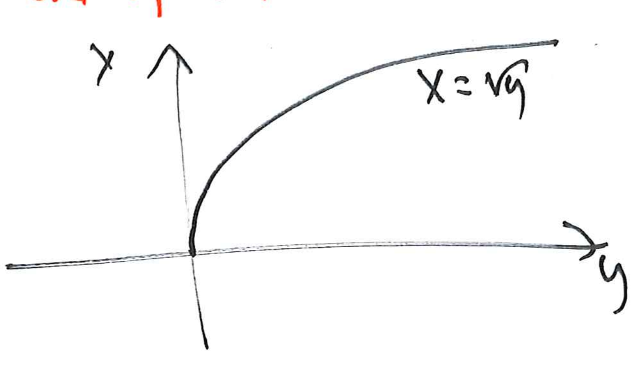
so $f^{-1}(2+\pi) = \pi$ ("the inverse image of $2+\pi$ is π")

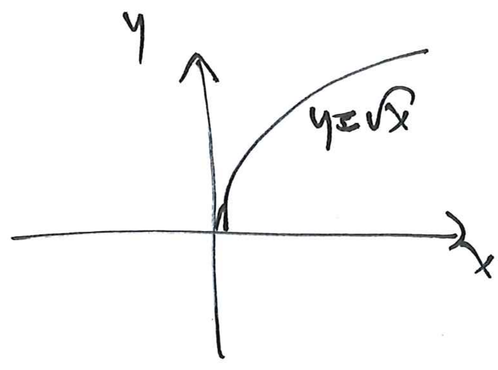
Example: $y = x^2$





no inverse function, but we force one by choosing one of the branches

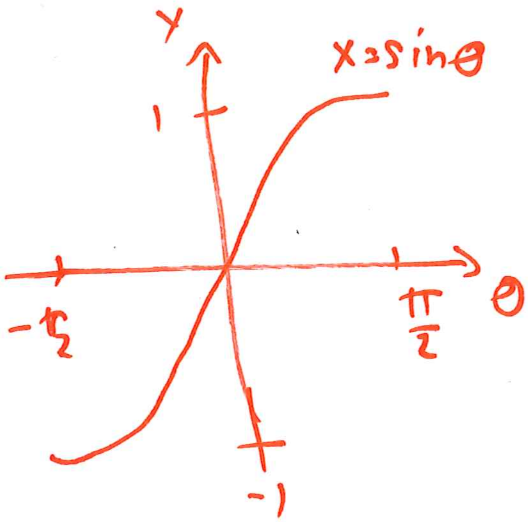


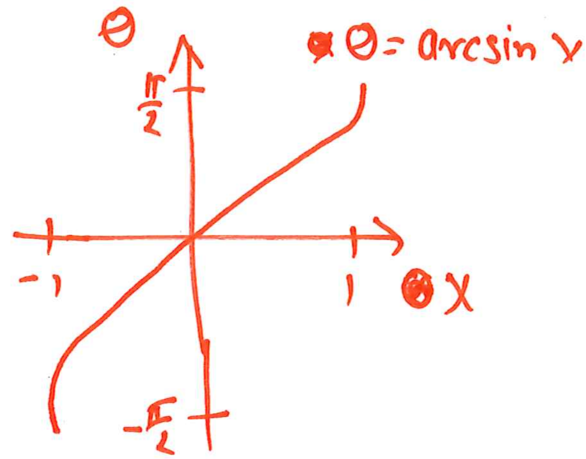


Some inverse functions

(1) $\theta = \arcsin x$, if $\sin\theta = x$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

On $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin\theta$ takes every value in $[-1,1]$ exactly once.





(2) $y = \log x$ if $x = e^y$.

(also arccos, arctan)

Remark about domains for inverse functions:

(1) domain of $\arcsin x$, is those x for which $\sin\theta = x$ has a solution, i.e. $x \in [-1,1]$

(arccos same, but arctan defined everywhere)

(2) domain of $\log x$ = range of e^x, i.e. $(0,\infty)$

Note: $\sin^{-1} x$ is another notation for $\arcsin x$,

but note: $\sin^{-1} x \neq \frac{1}{\sin x} \neq \arcsin x$

(3) Consider the function $y = \sqrt{x-1}$ on $x \geq 1$.

(a) Find the inverse function, in the form $x = g(y)$.

Solving for x, we find $x = y^2 + 1$

(b) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$ and calculate their product.

$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}} \cdot 1 = \frac{1}{2\sqrt{x-1}}$; $\frac{dx}{dy} = 2y$

so $\frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{\cancel{2}y}{\cancel{2}\cancel{\sqrt{x-1}}} = 1$ along our curve, $y = \sqrt{x-1}$

3. The inverse function fule

(1) Given that $\frac{d}{dy}y^2 = 2y$, find $\frac{d}{dx}\sqrt{x}$.

if $y = \sqrt{x}$, $x = y^2$, so $\frac{dx}{dy} = 2y$ so $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$

invert

(2) Find $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x$.

If $\theta = \arcsin x$, θ is the angle such that $\sin\theta = x$

so $\frac{dx}{d\theta} = \cos\theta$ and $\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$.

$\cos^2\theta + \sin^2\theta = 1$

(Also true that $\frac{d\theta}{dx} = \frac{1}{\cos(\arcsin x)}$)

(3) Find $\frac{\mathrm{d}}{\mathrm{d}x} \log x$.

If $y = \log x$, then $x = e^y$, so $\frac{dx}{dy} = e^y$, so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Try $\frac{d}{dx} \log_{10} x$.

If $y = \log_{10} x$, $x = 10^y$, so $\frac{dx}{dy} = (\log 10) \cdot 10^y$

so $\frac{dy}{dx} = \frac{1}{(\log 10) \cdot 10^y} = \frac{1}{(\log 10) \cdot x}$

(4) (Derivatives and logarithms)

(a) Differentiate $\log \sqrt[k]{t}$.

Solution 1: $\frac{d}{dt}(\log \sqrt[k]{t}) = \frac{1}{\sqrt[k]{t}} \cdot t^{\frac{1}{k}-1} \cdot \frac{1}{k} = \frac{1}{k} \cdot t^{1/k - 1} / t^{1/k} = \frac{1}{kt}$

Solution 2: Set $y = \log \sqrt[k]{t}$ then $e^y = \sqrt[k]{t}$ so $t = (e^y)^k = e^{ky}$

So $\frac{dt}{dy} = ke^{ky}$ so $\frac{dy}{dt} = \frac{1}{ke^{ky}} = \frac{1}{kt}$

Solution 3: $\log t^{1/k} = \frac{1}{k} \log t$ so $\frac{d}{dt}(\cancel{\log}\, t^{1/k}) = \frac{d}{dt}\left(\frac{1}{k}\log t\right) = \frac{1}{k} \cdot \frac{1}{t} = \frac{1}{kt}$

(b) (Final, 2012) Let $y = \log(\sin(\log x))$. Find $\frac{dy}{dx}$.

By the chain rule,

$$\frac{dy}{dx} = \frac{1}{\sin(\log x)} \cdot \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x \sin(\log x)}$$