10. Implicit Differentiation; Inverse Trig (8/10/2019)

Goals.

(1) Implicit differentiation
(2) Evaluating inverse trig functions
(3) Differentiating inverse trig functions

Last Time.

Logarithms; (1) solve equation $N_0 e^{kt} = N$ for t by taking logarithm

(2) $\frac{d}{dx}\log|x| = \frac{1}{x}$ $(x \neq 0)$ ← if $x > 0$, $y = \log x \iff x = e^y$ so $1 = e^y y' = x y'$ so $y' = \frac{1}{x}$

(3) $\frac{df}{dx} = f \cdot \frac{d(\log f)}{dx}$.

An explicitly defined function is one given as $y = f(x)$

An implicitly defined function is one given by a rule $F(x,y) = 0$

Example: consider curve $e^{x+y} + \cos x + \sin y = 0$

(if $x = 0$, y value is the solution to $e^y + 1 + \sin y = 0$)

Example: logarithm was defined as the solution to

$x = e^y$ or $e^y - x = 0$

Problem: suppose y is given as an implicit function of x. Can we compute $\frac{dy}{dx}$?

Solution: If $F(x,y)=0$ then it's derivative wrt x is 0 too, can use chain rule to isolate $\frac{dy}{dx}$.

Example: if $e^{x+y}+\cos x+\sin y=0$

then $\frac{d}{dx}\left(e^{x+y}+\cos x+\sin y\right)=0$

"diff. along the curve"

so: $e^{x+y}(1+y') - \sin x + (\cos y)\, y' = 0$

linear equation in unknown y'

$$\Leftrightarrow \quad (e^{x+y}+\cos y)\, y' = \sin x - e^{x+y}$$

so
$$y' = \frac{\sin x - e^{x+y}}{e^{x+y}+\cos y}$$

Math 100 – WORKSHEET 10
IMPLICIT DIFFERENTIATION; INVERSE TRIG FUNCTIONS

1. Implicit Differentiation

(1) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.

Differentiating along the curve, we find $2y \cdot y' = 12x^2 + 2$

so $y' = \frac{6x^2+1}{y}$ so slope at $(2,6)$ is $\frac{6 \cdot 2^2 + 1}{6} = \frac{25}{6}$, and the line is: $\boxed{y = \frac{25}{6}(x-2) + 6}$

(2) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

~~Here~~ Differentiating, we get: $y^2 + x \cdot 2y \cdot y' + 2xy + x^2 \cdot y' = 0$

i.e. $1 + 2y' + 2 + y' = 0$, i.e. $3y' = -3$, i.e. $\boxed{\frac{dy}{dx}\Big|_{(1,1)} = -1}$

(3) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

Along the curve we have: $y' + \cos y - x(\sin y)y' = -\sin x$

At $(0,1)$ this reads: $y' + \cos 1 - 0 = 0$ so $\boxed{y' = -\cos 1}$

Date: 8/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(4) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

First, we have $5x^4 + 5y^4y' = 0$ along the curve i.e.

$$y' = -\frac{x^4}{y^4}.$$

So $y'' = -\frac{4x^3}{y^4} + \frac{4x^4}{y^5}y' = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9} = -4\frac{x^3y^5 + x^8}{y^9}.$

Alternative: have $\cancel{5}x^4 + \cancel{5}y^4y' = 0$, diff again!

get: $4x^3 + 4y^3(y')^2 + y^4y'' = 0$

so $y'' = -4\frac{x^3 + y^3(y')^2}{y^4} = -4\frac{x^3 + y^3\frac{x^8}{y^8}}{y^4}.$

(5) Find y' if $(x + y)\sin(xy) = x^2$.

Same works for $\arccos(\cos\theta)$. (except $\cos(\pi-\theta) = -\cos\theta$)

Easy for arctan (since $\tan\theta$ has period π)

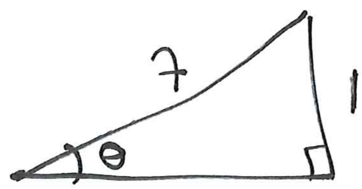
What about $\sin(\arcsin x)$? $\sin(\arcsin x) = x$ by def'n

What about $\cos(\arcsin x)$?

Example: Want $\cos(\arcsin \frac{1}{7})$.

Solution: Draw triangle

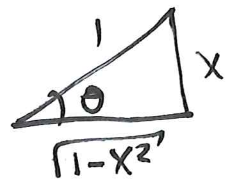
$\sin\theta = \frac{1}{7}$



(choose two sides to make $\sin\theta = \frac{1}{7}$)

By Pythagoras, third side is $\sqrt{48}$, so $\cos\theta = \frac{\sqrt{48}}{7}$.

in general: $\cos(\arcsin x) = \sqrt{1-x^2}$



$\sin\theta = x$ if side x, hyp. $= 1$

Worksheet §2, 1(b), 1(c)

Inverse trig functions

Recall: $\theta = \arcsin x$ if $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin\theta = x$

$\theta = \arccos x$ if $\theta \in [0, \pi]$ and $\cos\theta = x$

$\theta = \arctan x$ if $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\tan\theta = x$

Examples: $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$, $\arccos(\frac{1}{2}) = \frac{\pi}{3}$

(note, $\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$)

$\arcsin(\sin(4\pi + \frac{\pi}{6})) = ?$

(1) $\sin(4\pi + \frac{\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$ so this is $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$

(2) since $-\frac{\pi}{2} \le \frac{\pi}{6} \le \frac{\pi}{2}$, and $\sin(\frac{\pi}{6}) = \sin(4\pi + \frac{\pi}{6})$,

$\arcsin(4\pi + \frac{\pi}{6}) = \frac{\pi}{6}$.

For same reason, $\arcsin(\sin(4\pi + \frac{\pi}{7})) = \frac{\pi}{7}$

Warning: $\arcsin(\sin\theta))$ not always θ ($\sqrt{x^2}$ not always x)

To find $\arcsin(\sin\theta)$, use periodicity to put $-\pi \le \theta \le \pi$ and $\sin(\pi - \theta) = \sin\theta$

Example: $\sin(\frac{26\pi}{7}) = \sin(\frac{26\pi}{7} - 4\pi) = \sin(-\frac{2\pi}{7})$

so $\arcsin(\sin\frac{26\pi}{7}) = -\frac{2\pi}{7}$

Worksheet Sec 2, 1(a)

2. Inverse trig functions

(1) Evaluation

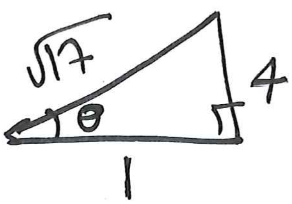
(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right) = -\frac{\pi}{6}$. Also, $\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31}{11}\pi - 2\pi\right) =$

$= \sin\left(\frac{9\pi}{11}\right) = \sin\left(\pi - \frac{9\pi}{11}\right) = \sin\left(\frac{2}{11}\pi\right)$ so

$\arcsin\left(\sin\left(\frac{31}{11}\pi\right)\right) = \arcsin\left(\sin\left(\frac{2\pi}{11}\right)\right) = \frac{2\pi}{11}$

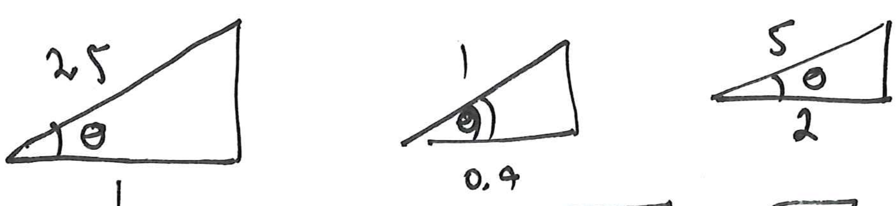
(b) (Final 2015) Simplify $\sin(\arctan 4)$



this triangle has $\tan\theta = 4$, $\sin\theta = \frac{4}{\sqrt{17}}$.

so $\sin(\arctan 4) = \frac{4}{\sqrt{17}}$

(c) Find $\tan(\arccos(0.4))$



in any case we find $\tan\theta = \frac{\sqrt{1-(0.4)^2}}{0.4} = \frac{\sqrt{0.84}}{0.4} = \frac{\sqrt{84}}{4} = \sqrt{\frac{21}{4}}$.

Differentiating inverse trig

Already saw: $$\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

since arcsin x + arccos x = $\frac{\pi}{2}$, get

$$\frac{d(\arccos x)}{dx} = -\frac{d(\arcsin x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Also, $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$ (because $(\tan\theta)' = 1 + \tan^2\theta$)

Warning: $\arcsin x \neq \frac{1}{\sin x}$

$y = \arcsin x$ if $x = \sin y$

$y = \frac{1}{\sin x}$ if $y \cdot \sin x = 1$

(2) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$

This is $\frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$

(b) Find the line tangent to $y = \sqrt{1+(\arctan(x))^2}$ at the point where $x = 1$.

$\frac{dy}{dx} = \frac{1}{2\sqrt{1+(\arctan x)^2}} \cdot 2\arctan x \cdot \frac{1}{1+x^2} = \frac{\arctan x}{\sqrt{1+(\arctan x)^2}} \cdot \frac{1}{1+x^2}$

at $x=1$, the slope is $\frac{\pi/4}{\sqrt{1+\frac{\pi^2}{16}}} \cdot \frac{1}{1+1} = \frac{\pi}{2\sqrt{16+\pi^2}}$

so the line is $y = \frac{\pi}{2\sqrt{16+\pi^2}}(x-1) + \sqrt{1+\frac{\pi^2}{16}}$.

(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?