12. Exponetial growth and decay (15/10/2019)

Goals.
(1) Exponetial growth
(2) Half-life
(3) Newto's law of cooling

Midterm
Thu, in-class ~50min
~similar to sample exam
*follow instructions

Last Time.
Applications of the derivative: (1) velocity = rate of change of position
(2) acceleration = " " " " velocity
(3) speed

More generally, physical quantities can be derivatives of other quantities

End of class: students arrive to office hours independently.
prob of seeing a student in short time interval is $\approx r \cdot dt$.

Calc: If $P(t)$ prob of waiting t without seeing anyone

then $P'(t) = -rP(t)$.

Remarks: (1) This kind of object is called a differential equation
(it's an equation, involving derivatives, unknown is a function)
(2) This particular equation is common.

Solving this equation

Note: $(e^t)' = e^t$, so $(e^{kt})' = ke^{kt}$.

Also (linearity) $(Ce^t)' = Ce^t$.

similarly $(Ce^{kt})' = k(Ce^{kt})$.

Conclusion: The function $y(t) = Ce^{kt}$ solves the diff eqn

$$\frac{dy}{dt} = ky.$$

(eg.) Eg. if $P'(t) = -rP(t)$, then $P(t) = Ce^{-rt}$.

(the probability of waiting time t decays exponentially at rate r)

Can find C by considering $P(0) = 1$

$\Rightarrow C = 1 \quad (e^{-0} = 1)$

– if know r, can find C from $P(t_0)$ at any time t_0.

Math 100 – WORKSHEET 12
EXPONENTIAL GROWTH AND DECAY

1. Exponentials

(1) Suppose[1] that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.

(a) At what time will there be 1000 opossums in BC? 10,000 opossums?

Let $N(t)$ be the number of oppossums, t years after 1935

Then $N(t) = 2 \cdot 3^t = 2(e^{\log 3})^t = 2 \cdot e^{(\log 3) \cdot t}$ Then

$N(t) = 1000$ if $2 \cdot 3^t = 1000$, so $3^t = 500$ so $t = \frac{\log 500}{\log 3}$

Similarly $N(t) = 10{,}000$ if $t = \frac{\log 5{,}000}{\log 3}$

(b) Write a differential equation expressing the growth of the opossum population with time.

Sol'n 1: exp. growth rate is $\log 3 \Rightarrow$ eqn is $\frac{dN}{dt} = (\log 3) \cdot N$

Sol'n 2: $\frac{dN}{dt} = \frac{d}{dt}(2 \cdot 3^t) = 2 \cdot 3^t \cdot \log 3 = (\log 3) \cdot \frac{dN}{dt}$

Date: 15/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.
[1]See http://linnet.geog.ubc.ca/efauna/Atlas/Atlas.aspx?sciname=Didelphis%20virginiana

Common to measure exponential growth rate in terms of doubling time or decay in terms of halving time commonly called the half-life

(i.e. instead of writing $C \cdot e^{kt}$, we write

$$C \cdot 2^{t/\tau} \quad \text{or} \quad C \cdot 2^{-t/\tau} = C \cdot \left(\tfrac{1}{2}\right)^{t/\tau}$$

where τ = doubling time / half-life.

Note: $2^{t/\tau} = e^{\frac{\log 2}{\tau} \cdot t}$. (can convert from half-life/doubling time to exp growth rate and back)

(2) A radioactive sample decays according to the law
$$\frac{dm}{dt} = km\,.$$

(a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?

(b) A 100-gram sample is left unattended for three days. How much of it remains?

(a) $\frac{1}{4} = \left(\frac{1}{2}\right)^2$, so two halvings take 10 hours, and $t_{1/2} = \tau = 5h$

(b) $m(3\text{days}) = \underbrace{(100\,gr)}_{\text{quantity at } t=0} \cdot 2^{\left(-\frac{72}{5}\right)}$.

$$= 100 \cdot \left(\frac{1}{2}\right)^{\frac{72}{5}}\ gr$$

$$= 100 \cdot e^{-\log 2 \cdot \frac{72}{5}}.$$

(3) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.

The initial number is $\frac{2000}{2^{1.5}}$ ← 1.5 doublings in 6 hours $= \frac{1000}{\sqrt{2}} \approx 707$

Alternative: $N(t) = N_0 \cdot 2^{t/4}$, t in hours so $2000 = N_0 \cdot 2^{6/4}$

number at initial

so $N_0 = \frac{2000}{2^{1.5}}$

2. Newton's Law of Cooling

Fact. *When a body of temperature T_0 is placed in an environment of temperature T_{env}, the rate of change of the temperature $T(t)$ is* negatively proportional *to the temperature difference $T - T_{env}$. In other words, there is a (negative) constant k such that*

$$T' = k(T - T_{env}).$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\mathrm{env}}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = ky$$

so there is C for which

$$y(t) = Ce^{kt}. \leftarrow$$ the temperature difference decays exponentially

Solving for T we get:

$$T(t) = T_{\mathrm{env}} + Ce^{kt}.$$

Setting $t = 0$ we find $T_{\mathrm{env}} + C = T_0$ so $C = T_0 - T_{\mathrm{env}}$ and

$$T(t) = T_{\mathrm{env}} + (T_0 - T_{\mathrm{env}})e^{kt}.$$

Corollary. $\lim_{t\to\infty} y(t) = 0$ *so* $\lim_{t\to\infty} T(y) = T_{env}$.

(1) (Final, 2010) When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.

Let $y(t)$ be the difference in temperature between the apple and the room, where t is measured in minutes. We are told that $y(0) = -16°C$, $y(30) = 11-19 = -8°C$, since $y(30) = \frac{1}{2} y(0)$, and Newton's law of cooling says $y(t)$ decays exponentially, we see that $y(t) = y(0) \cdot \left(\frac{1}{2}\right)^{t/30}$

so $y(90) = -16°C \cdot \frac{1}{8} = -2°C$, so $T(90) = -2°C + 19°C = 17°C$

(b) Determine the time when the temperature of the apple is $16°C$.

We need t so that $-16°C \cdot \left(\frac{1}{2}\right)^{t/30} = -3°C$, i.e. ← $16°C - 19°C$

$$\left(\frac{1}{2}\right)^{t/30} = \frac{3}{16} \quad \text{so} \quad \frac{t}{30} \log\left(\frac{1}{2}\right) = \log 3 - \log 16$$

$$\text{so} \quad t = \frac{\log 16 - \log 3}{\log 2} \cdot 30 \text{ minutes.}$$

(c) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.

Decay rate is $\left(\frac{1}{2}\right)^{t/30} = e^{-\frac{\log 2}{30} t}$ so the DE is

$$\frac{dT}{dt} = -\frac{\log 2}{30}(T - 19°C).$$