14. Taylor expansion (22/10/2019)

Goals.
(1) Linear approximation
(2) Higher-order approximation
(3) Combining expansions

Last Time.

Midterm: (1) grades on Canvas (2) email: how to retrieve papers
- questions about midterm, advice on studying, ... :
  office hours: Thu after class, Mon 12:30-14:00 at IBL.
- regrades: contact instructor-in-charge, Prof. Bennett.

Related rates: can diff. a relationship between x,y wrt t.
Creates equation connecting $x, y, \frac{dx}{dt}, \frac{dy}{dt}$, can solve for one given other three (sometimes need only 2, if also use original equation)

(Contrast with implicit diff: same relationship, by diff wrt x, so get equation connecting $x, y, \frac{dy}{dx}$)

Taylor expansion: Idea: to evaluate a complicated function using only basic arithmetic. Achieved by approximating the function by a polynomial

Problem: Have function $f(x)$, want to evaluate it.

Work near point $x=a$, extrapolate from $x=a$ to x close to a

0^{th} order approximation

If f is cts, and x is close to a then $f(x)$ is close to $f(a)$

("$\log 1.1 \approx \log 1 = 0$", "$\sin\left(\frac{\pi}{2} - \frac{1}{10}\right) \approx \sin\frac{\pi}{2} = 1$")

1^{st} order approximation

Try to approx f near a by adding a linear correction:

want to say $f(x) \approx f(a) + m \cdot (x-a)$.

This will best approximate f if they have same slope,

i.e. if we take $m = f'(a)$:

the linear approximation to f near a is

$$f(x) \approx f(a) + f'(a)(x-a)$$

$\approx$ = "approximately equal to"

Worksheet (1)

Summary: compute tangent line, evaluate at x close to a

Warning: never do $f(x) \approx f(a) + f'(x)(x-a)$

e.g. $\sqrt{x} \approx \sqrt{1} + \frac{1}{2\sqrt{x}}(x-1)$

Math 100 – WORKSHEET 14
TAYLOR EXPANSION

$\sqrt{8} = 2.828\ldots$

1. The Linear Approximation

(1) Use a linear approximation to estimate

(a) $\sqrt{1.2}$

Let $f(x) = \sqrt{x}$, we want $f(1.2)$. We have $f(1) = \sqrt{1} = 1$

$\frac{df}{dx} = \frac{1}{2\sqrt{x}}$, so $f'(1) = \frac{1}{2}$, so linear approx is $f(1.2) \approx f(1) + f'(1)\cdot(1.2-1)$
$= 1 + \frac{1}{2}0.2 = 1.1$

(can also try $a = 1.21$)

(b) (Final, 2015) $\sqrt{8}$

Let $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, so $f(9) = 3$, $f'(9) = \frac{1}{6}$, so to first order,

$f(8) \approx f(9) + f'(9)\cdot(-1) = 3 - \frac{1}{6} = 2\frac{5}{6}$

(c) (Final, 2016) $(26)^{1/3}$

Let $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3x^{2/3}}$. So $f(27) = 3$, $f'(27) = \frac{1}{27}$, and to first order $f(26) \approx f(27) + f'(27)\cdot(26-27) \approx 3 - \frac{1}{27} \approx 2\frac{26}{27}$

(d) $\log 1.07$

See: to first order $\log(1+x) \approx x$, so $\log(1.07) \approx 0.07$.

Date: 22/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Taylor approximation

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(a) $f'(x) = e^x$, $f^{(2)}(x) = e^x$, $f^{(3)}(x) = e^x$, ..

2nd deriv. 3rd deriv. ..

so $f(0) = 1$, $f'(0) = 1$, $f^{(2)}(0) = 1$, ..

(b) $T_0(x) = 1 \leftarrow f(0)$ (c) $T_1(x) = 1 + x$

(d) try $T_2(x) = 1 + x + cx^2$, get $T_2''(x) = 2c$, so choose $c = \frac{1}{2}$ to get $T_2''(0) = 1$, i.e. $T_2(x) = 1 + x + \frac{1}{2}x^2$

"quadratic correction to linear approx"

(e) Take $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ | quadratic approx

cubic approx to e^x about $x=0$

cubic correction / cubic term

(3) Do the same with $f(x) = \ln x$ about $x = 1$.

$f^{(1)}(x) = \frac{1}{x}$, $f^{(2)}(x) = -\frac{1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$

so $f(1) = 0$, $f'(1) = 1$, $f^{(2)}(1) = -1$, $f^{(3)}(1) = 2$

so $T_0(x) = 0$, $T_1(x) = 1 \cdot (x-1)$, $T_2(x) = x - 1 + \left(-\frac{1}{2}\right)(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$

$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

Procedure: to add k'th term, we need $\_\_\,(x-a)^k$ which matches k'th derivative. But k'th derivative of $(x-a)^k$ is: $k(k-1)(k-2)\cdots 2\cdot 1 = k!$ so need coeff $\frac{f^{(k)}(a)}{k!}(x-a)^k$

memorize

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$) let $g(x) = \frac{1}{1-x}$, Then

$g'(x) = \frac{1}{(1-x)^2}$, $g^{(2)}(x) = \frac{1\cdot 2}{(1-x)^3}$, $g^{(3)}(x) = \frac{1\cdot 2\cdot 3}{(1-x)^4}$, $g^{(4)}(x) = \frac{1\cdot 2\cdot 3\cdot 4}{(1-x)^5}$

So $g(0) = 1$, $g'(0) = 1$, $g^{(2)}(0) = 1\cdot 2$, $g^{(3)}(0) = 1\cdot 2\cdot 3$, $g^{(4)}(0) = 1\cdot 2\cdot 3\cdot 4$

So $T_4(x) = 1 + \frac{1}{1!}x + \frac{1\cdot 2}{2!}x^2 + \frac{1\cdot 2\cdot 3}{3!}x^3 + \frac{1\cdot 2\cdot 3\cdot 4}{4!}x^4 = 1 + x + x^2 + x^3 + x^4$

(5) Find the nth order expansion of $\cos x$.

In order, the derivatives of $\cos x$ are: $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, ...

So the values at $x = 0$: 1, 0, −1, 0, 1, 0, −1, 0, 1, 0, −1, 0, ..

so the ~~exp~~ Taylor expansion is:

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \ldots$$

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$? we know coeff of $(x-3)^2$ is $\frac{f^{(2)}(3)}{2!} = 12$

So $f^{(2)}(3) = 24$.