17. The Mean Value Theorem (31/10/2019)

Goals.

(1) More minima and maxima
(2) The Mean Value Theorem

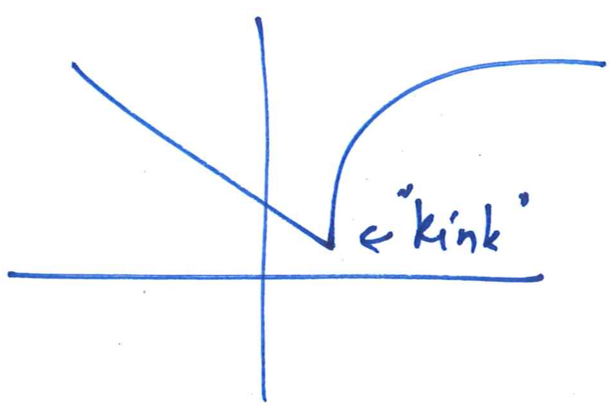
Last Time.

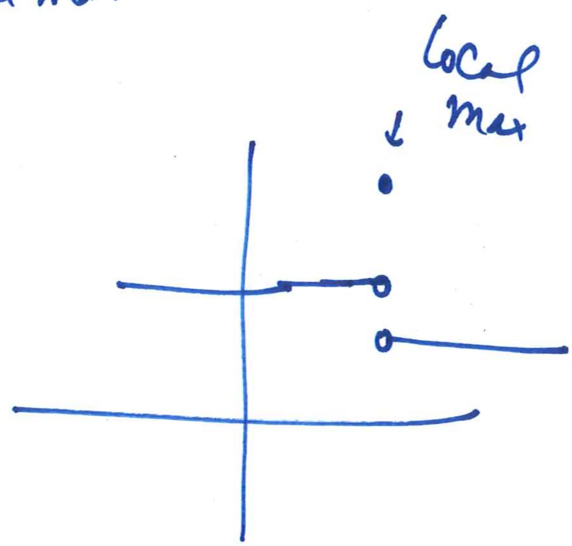
Minima & maxima of functions:

(1) global/absolute maxima (e.g. f cts on closed interval)

(2) local extrema - occur at critical/singular pts

The following are local extrema:





Math 100 – WORKSHEET 17
THE MEAN VALUE THEOREM

1. More minima and maxima

(1) Show that the function $f(x) = 3x^3 + 2x - 1 + \sin x$ has no local maxima or minima. You may use that $f'(x) = 9x^2 + 2 + \cos x$.

(1) f is everywhere diff so any local extremum would be a critical point.

(2) $f'(x) = 9x^2 + 2 + \cos x \geq 0 + 2 - 1 = 1 > 0$ so no crit. pts.

(2) Let $g(x) = xe^{-x^2/8}$ so that $g'(x) = \left(1 - \frac{x^2}{4}\right) e^{-x^2/8}$, find the global minimum and maximum of g on

(a) $[-1, 4]$ (b) $[0, \infty)$

* g is everywhere diff. It has critical pts at: $x = \pm 2$

(a) on $[-1,4]$, $g(-1) = -e^{-1/8}$, $g(2) = 2e^{-1/2}$, $g(4) = 4e^{-2}$

$g(-1)$ smallest (only negative value of the three) so

$\min_{x \in [-1,4]} f(x) = -e^{-1/8}$. Also, $g(2) = \frac{2}{\sqrt{e}} > \frac{2}{\sqrt{4}} = 1 = \frac{4}{2^2} > \frac{4}{e^2}$.

$e < 4$ $\quad$ $e > 2$

so $\max_{x \in [-1,4]} f(x) = \frac{2}{\sqrt{e}}$.

Date: 1/11/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) $g(0) = 0 \cdot e^0 = 0$ but if $x > 0$, $g(x) = x \cdot e^{-x^2/8} > 0$

since $e^y > 0$ for all y. So 0 is the minimum value

on $[0, \infty]$

We know $g(2) = \frac{2}{\sqrt{e}}$, and g has a local max there

this the global max because g is increasing

on $[0, 2]$, ($g' > 0$ on $(0, 2)$),

decreasing on $[2, \infty)$ ($g' < 0$ on $(2, \infty)$)

(alternative: $\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x}{e^{x^2/8}} = 0$ (pf at end of course)

so max of g must occur at some finite x, so must be a critical pt.

Mean Value Theorem

If f is cts $[a,b]$, diff on (a,b)

then there exists c, $a<c<b$ s.t. slope at c = average slope on $[a,b]$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

rearrage (call $b = $ "x") to see: given a, x there is c between a, x s.t.

$$f(x) = f(a) + f'(c)(b-a)$$

Worksheet (5), (6), (7)

Observe: If f diff, between two zeroes of f is a zero of f'! (need to use MVT to justify this on exams)

Worksheet (8): If f has three zeroes $\Rightarrow f''$ has a zero

to rule out three zeroes, enough to show f'' has no zeroes

(3) Find the critical numbers and singularities of $h(x) = \begin{cases} x^3 - 6x^2 + 3x & x \le 3 \\ \sin(2\pi x) - 18 & x \ge 3 \end{cases}$

for $x<3$, $h'(x) = 3x^2 - 12x + 3 = 3(x^2 - 4x + 1)$.
(vanishes at $\frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$, i.e. at $2-\sqrt{3}$ is a critical #. For $x>3$, $h'(x) = 2\pi\cos(2\pi x)$, vanishes at $3\frac{1}{4}, 3\frac{3}{4}, 4\frac{1}{4}, 4\frac{3}{4}, \ldots$ Also, 3 is a singularity:

(4) (Final, 2014) Find a such that $f(x) = \sin(ax) - x^2 + 2x + 3$ has a critical point at $x = 0$.

$$\lim_{h\to 0^-} \frac{h(3+h)-h(3)}{h} = \frac{d}{dx}\Big|_{x=3} (x^3 - 6x^2 + 3x) = [3x^2 - 12x + 3]_{x=3} = -6$$

$$\lim_{h\to 0^+} \frac{h(3+h)-h(3)}{h} = \frac{d}{dx}\Big|_{x=3} (\sin(2\pi x) - 18) = 2\pi\cos(2\pi\cdot 3) = 2\pi \neq 6$$

2. Average slope vs Instantenous slope

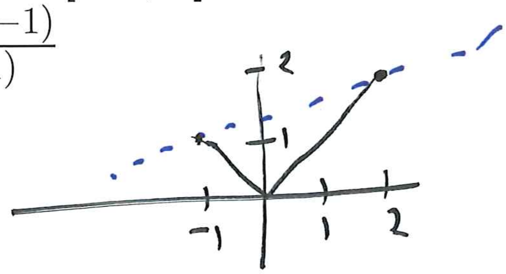
(5) Let $f(x) = e^x$ on the interval $[0, 1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

Need to find c s.t. $e^c = \frac{e-1}{1}$, i.e. $c = \log(e-1)$

(6) Let $f(x) = |x|$ on the interval $[-1, 2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$

Now need $f'(c) = \frac{1}{3}$, but

$f'(c) = \begin{cases} 1 & c>0 \\ -1 & c<0 \end{cases} \Rightarrow$ No such c!



3. The Mean Value Theorem

(7) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero. (Hint: let a, b be zeroes of f. The MVT will find c such that $f'(c) = ?$)

f is everywhere diff. Suppose $a<b$ are two different zeroes of f, then by MVT there is c between a, b s.t.
$f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. But (see problem (1)), $f' \neq 0$ everywhere

To see f has at least one zero, use IVT

(8) (Final, 2015)

(a) Suppose f, f', f'' are all continuous. Suppose f has at least three zeroes. How many zeroes must f', f'' have?

[Justify: between two zeroes of f have zero of f'] $\Rightarrow$ f' has at least two zeroes, so f'' has at least one zero.

(b) [Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions]

(c) Show that the equation has at most two solutions.

Let $f(x) = 2x^2 - 3 + \sin x + \cos x$. Then $f''(x) = 4 - \sin x - \cos x$.
So, for any x, $f''(x) \geq 4 - 1 - 1 \geq 2 > 0$ so f'' never vanishes
$\sin x, \cos x \leq 1$

By part (a), f ~~[illegible]~~ cannot have 3 or more zeroes, so it has at most 2.

(9) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1,4]$. What can you say about $f(4)$?

Since f' exists on $[1,4]$, the MVT applies and says: there is $c \in (1,4)$ so that

$$\frac{f(4)-f(1)}{4-1} = f'(c)$$

so $-3 \leq \dfrac{f(4)-f(1)}{4-1} \leq 2$ so $\underbrace{f(1)-3\cdot 3}_{=-6} \leq f(4) \leq f(1) + 2\cdot 3 = 9$

(10) Show that $|\sin a - \sin b| \leq |a-b|$ for all a, b.

(11) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1+x) \leq x$.

Question: show that $f(x) = 2x^2 - 3 + \sin x + \cos x$ has two zeroes ~~on~~ at least.

Solution: f is cts everywhere (def by formula)

$$f(10) = 2\cdot 10^2 - 3 + \sin 10 + \cos 10 \geq 197 - 1 - 1 = 195 > 0$$

$$f(-10) = 2(-10)^2 - 3 - \sin 10 + \cos 10 \geq 200 - 3 - 1 - 1 = 195 > 0$$

$$f(0) = -3 + \sin 0 + \cos 0 = -2 < 0$$

By IVT f has a zero between $(-10, 0)$, another one between $(0, 10)$