21. Optimization (14/11/2019)

Goals.

(1) Problem solving
(2) Examples

Last Time.

Review.

Question: Can a function increase on at interval where $f'(x)=0$ occassionally?

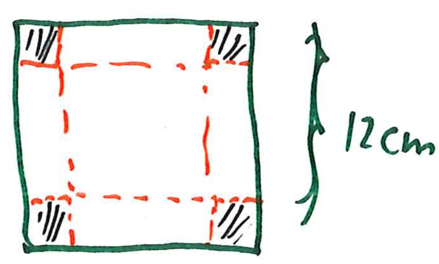
Example: $f(x)=x^3$ strictly increases on $\mathbb{R}$, but $f'(0)=0$.
- by first derivative test, f increases on $(-\infty,0)$, on $(0,\infty)$.
- Also, f cts at $x=0$, so f increases through 0.

Optimization: ~~Strategy~~ Setup ⇝ calculus ⇝ answer

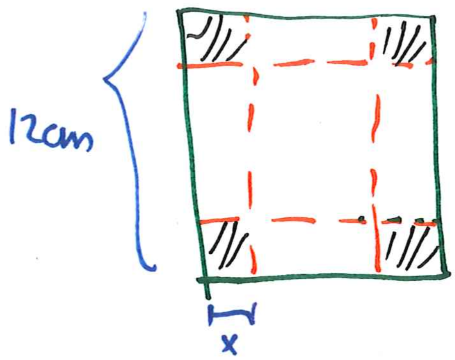
↑ today's focus ↑ "endgame"

Example: You are given a square sheet of cardboard (12cm × 12cm). Cutting small squares off the corners, we will fold the rest into a small box. What is the largest volume box constructed this way?

picture:



Let x be the side length (in cm) of the cut squares ($0 \le x \le 6$)



The base of the box will then be a square of side $(12-2x)$ cm

relation between quantities of interest

Let V be the volume (in cm^3) of the box. Then

$$V = (12-2x)^2 \cdot x$$

The problem is to find the maximum of $V(x)$ on the interval $[0,6]$. This function is continuous on $[0,6]$, $V(0) = V(6) = 0$, and V is positive in $(0,6)$ so its maximum will occur in the interior. $V(x)$ is everywhere diff, so the maximum will occur at a critical point.

$$V(x) = 4(6-x)^2 \cdot x \quad \text{so} \quad \frac{dV}{dx} = 8(6-x)(-1)x + 4(6-x)^2$$

$$= 4(6-x)(-2x + (6-x))$$

$$= 4(6-x)(6-3x) = 12(6-x)(2-x)$$

$$= 4(6-x)(6-3x) = 12(6-x)(2-x)$$

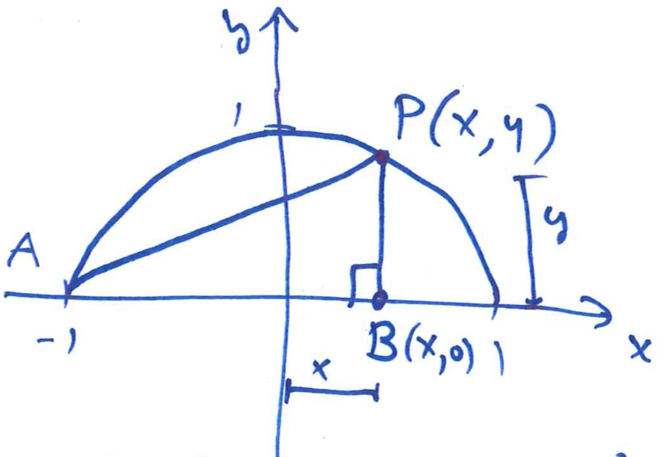
The only critical point is at $x=2$, where $V(2) = 128$ cm^3

We find that the largest volume box arises when we cut 2cm×2cm corners, and has volume 128 cm^3.

endgame

Math 100 – WORKSHEET 21
OPTIMIZATION

(1) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1-x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of this triangle?



Say P is at (x,y)
then B is at $(x,0)$
$(-1 \le x \le 1)$

The base of the triangle is the segment AB, of length $1+x$.
The height of the triangle is $y = \sqrt{1-x^2}$.
Thus, the area of the triangle is $A(x) = \frac{1}{2}(1+x)\sqrt{1-x^2}$
We need to find the max of $A(x)$ on $[-1,1]$. It's cts there.

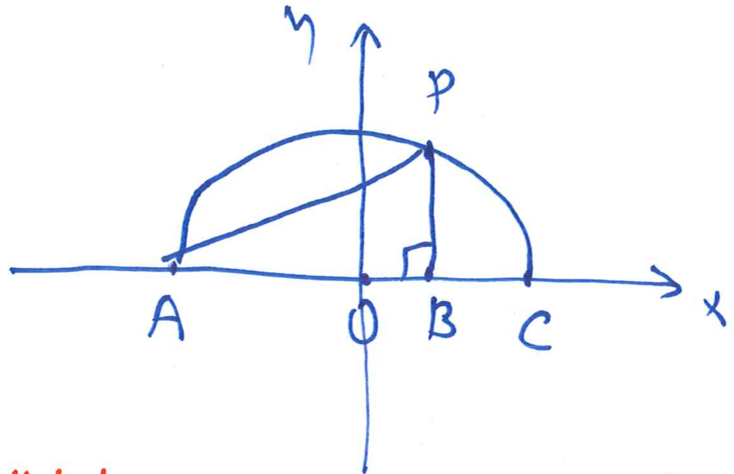
$$A'(x) = \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(1+x)\frac{-2x}{2\sqrt{1-x^2}} = \frac{1-x^2+(1+x)(-x)}{2\sqrt{1-x^2}} = \frac{1-2x^2-x}{2\sqrt{1-x^2}}$$

So $A'(x)$ exists in $(-1,1)$ where the critical points satisfy $2x^2+x-1=0$
i.e. $x = \frac{-1 \pm \sqrt{1+8}}{4} = -1, \frac{1}{2}$ so only critical pt is $x = \frac{1}{2}$.

Date: 14/11/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

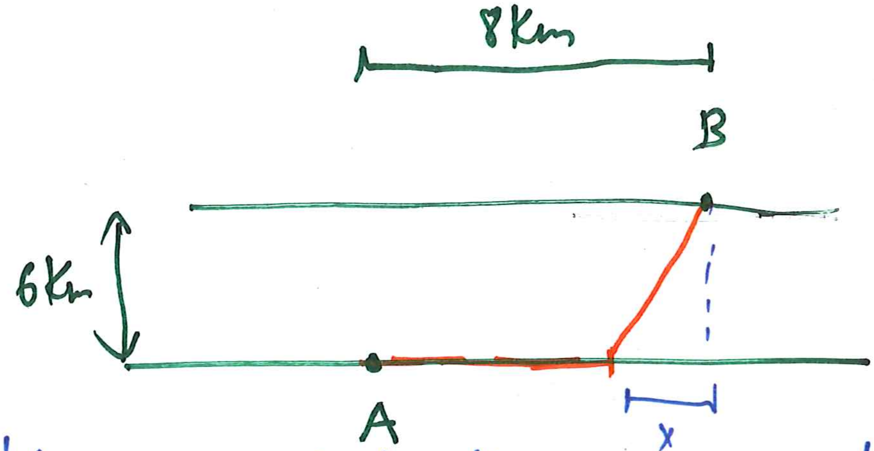
Now $A(-1) = A(1) = 0$, $A\left(\frac{1}{2}\right) = \frac{1}{2}\left(1+\frac{1}{2}\right)\sqrt{1-\frac{1}{4}} = \frac{3}{4}\sqrt{\frac{3}{4}} = \left(\frac{3}{4}\right)^{3/2}$.
So the largest possible area is $\left(\frac{3}{4}\right)^{3/2}$.

Suppose we're only given ~~"a right-angle~~ "a right-angled triangle has vertices ABP, where P is on a semicircle with diameter AC, and B (the right angle) lies on AC."



Say "let x axis pass on AC, with $A=(-r,0)$, $C=(r,0)$, r = radius of semicircle". Let y-axis point toward the semicircle so that if $P=(x,y)$, then $x^2+y^2=r^2$.

(2) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?



Suppose we build a road of length $8-x$ Km along the river, and a bridge of length $\sqrt{6^2+x^2}$ Km across it.
The cost of this construction is

$$C(x) = 40\sqrt{6^2+x^2} + 20(8-x)$$

This is a continuous function of x and its minimum certainly lies between $0 \le x \le 8$. C is diff there and

$$C'(x) = 40\frac{2x}{2\sqrt{6^2+x^2}} - 20,$$ so critical values satisfy

$$\frac{40x}{\sqrt{6^2+x^2}} = 20, \quad \text{so} \quad 2x = \sqrt{6^2+x^2} \quad \text{so} \quad 4x^2 = 36+x^2$$

so $x^2 = 12$, $x = \sqrt{12}$, and this is the only critical point in $[0,8]$.

$$C(0) = 40\cdot 6 + 20\cdot 8 = 400, \quad C(8) = 40\sqrt{6^2+8^2} = 400$$

$$C(\sqrt{12}) = 40\sqrt{36+12} + 160 - 20\sqrt{12} = 40\sqrt{48} + 160 - 20\sqrt{12}$$

Since $C'(0) = -20$, C goes below 400 near 0.

This means the minimum of C is below 400, so must be at the critical point.

So the most efficient way is to build a road of length $8-\sqrt{12}$ km, then bridge to B