Math 100 - Lecture 24 : Review 1

Plan: (1). Prepared examples

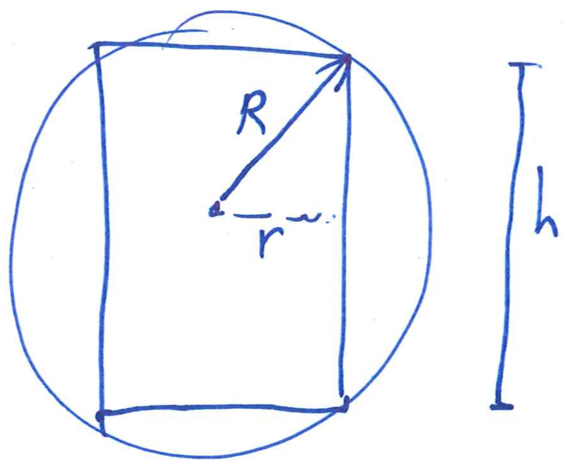
(2) open floor

Problem: (2014 final?) Of all cylinders inscribed in a sphere, find the one of largest volume. Say sphere has radius 6.

(0) understand problem

Solution: A cross section by a plane containing the axis of the cylinder is:

Let V be the volume of the cylinder, h its height, r its radius, $R=6$ the radius of the sphere.



step (1): name quantities

Since the radius of the cylinder is perpendicular to its side, we have $\left(\frac{1}{2}h\right)^2 + r^2 = R^2 = 6^2$. Also, $V = \pi r^2 h$

(2) enforce relations

Thus $V = \pi h\left(6^2 - \frac{h^2}{4}\right)$, or $V = \pi r^2 \cdot 2\sqrt{6^2 - r^2}$

where $0 \le h \le 2R = 12$

In summary, we need the maximum of $V(h) = \pi h\left(36 - \frac{h^2}{4}\right)$ on the interval $0 \le h \le 12$.

Now V is everywhere diff, $V'(h) = \pi\left(36 - \frac{h^2}{4} + h\cdot\left(-\frac{h}{2}\right)\right)$

$$= \pi\cdot\left(36 - \frac{3h^2}{4}\right).$$

this has critical points where $36 = \frac{3}{4}h^2$, ie $h^2 = 48$, i.e.

$$\boxed{h = 4\sqrt{3}}$$

Since $V(0) = 0$, $V(12) = \pi\cdot 12\cdot(36 - 6^2) = 0$

and $V(4\sqrt{3}) = \pi\cdot 4\sqrt{3}\left(36 - \frac{48}{4}\right) = 96\sqrt{3}\,\pi$.

(3) Calculus !

Thus the cylinder of largest volume has $h = 4\sqrt{3}$, and radius

$$r = \sqrt{6^2 - \frac{48}{4}} = \sqrt{24}.$$

(4) end game

Problem: Evaluate $\lim_{\theta \to \frac{\pi}{4}} 3(\tan(\theta))^{(\tan 2\theta)}$

Solution:

As $\theta \to \frac{\pi}{4}$, $\tan\theta \xrightarrow[\theta \to \frac{\pi}{4}]{} \tan\left(\frac{\pi}{4}\right) = 1$, but $\tan 2\theta \to \begin{cases} +\infty & \theta \to \frac{\pi}{4}^- \\ -\infty & \theta \to \frac{\pi}{4}^+ \end{cases}$

this is an indeterminate form (1^∞)

Consider $\log\left(3(\tan\theta)^{\tan(2\theta)}\right) = \log 3 + \log\left((\tan\theta)^{(\tan 2\theta)}\right) =$

$$= \log 3 + (\tan 2\theta)\cdot\log\tan\theta$$

not $(\tan 2\theta)\cdot\log(3\tan\theta)$, not $\tan\theta\log(\tan 2\theta)$.

not indeterminate:

$\tan 2\theta \to \pm\infty$, but $\log(3\tan\theta) \to \log 3$ as $\theta \to \frac{\pi}{4}$.

badly behaved if $\theta > \frac{\pi}{4}$

$$\lim_{\theta\to\frac{\pi}{4}} (\tan 2\theta)\cdot \log\tan\theta = \lim_{\theta\to\frac{\pi}{4}} \frac{\log\tan\theta}{1/\tan 2\theta} =$$

(or: $\lim_{\theta\to\frac{\pi}{4}} \tan(2\theta)\cdot\log\tan\theta = \lim_{\theta\to\frac{\pi}{4}} \frac{\sin 2\theta}{\cos 2\theta}\cdot\log\tan\theta =$

$$=\left(\lim_{\theta\to\frac{\pi}{4}} \sin 2\theta\right)\left(\lim_{\theta\to\frac{\pi}{4}} \frac{\log\tan\theta}{\cos 2\theta}\right) \Bigg)$$

$$\stackrel{LH}{=} \lim_{\theta\to\frac{\pi}{4}} \frac{\frac{1}{\tan\theta}\cdot(1+\tan^2\theta)}{-\frac{2}{\tan^2 2\theta}(1+\tan^2 2\theta)} = -\lim_{\theta\to\frac{\pi}{4}} \frac{(1+\tan^2\theta)\cdot\tan^2(2\theta)}{2\tan\theta\,(1+\tan^2 2\theta)}$$

$$\lim_{\theta\to\frac{\pi}{4}} \log\tan\theta = \log\tan\frac{\pi}{4} = 0$$

$$\lim_{\theta\to\frac{\pi}{4}} 1/\tan 2\theta = \lim_{\theta\to\frac{\pi}{2}} 1/\tan\theta = 0$$

$$= -\left(\lim_{\theta\to\frac{\pi}{4}} \frac{1+\tan^2\theta}{2\tan\theta}\right)\left(\lim_{\theta\to\frac{\pi}{4}} \frac{\tan^2(2\theta)}{1+\tan^2(2\theta)}\right)$$

look: $\frac{\tan^2(2\theta)}{1+\tan^2(2\theta)} = \frac{\sin^2(2\theta)/\cos^2(2\theta)}{1+\sin^2(2\theta)/\cos^2(2\theta)} = \frac{\sin^2(2\theta)}{\cos^2(2\theta)+\sin^2(2\theta)} = \sin^2(2\theta)$

So $\lim_{\theta\to\frac{\pi}{4}} (\tan 2\theta)\log\tan\theta = -\underbrace{\frac{1+1^2}{2\cdot 1}}_{\tan\frac{\pi}{4}=1}\cdot \underbrace{1^2}_{\sin(2\cdot\frac{\pi}{4})=1} = -1$

Endgame: So $\lim_{\theta\to\frac{\pi}{4}} 3(\tan\theta)^{\tan 2\theta} = 3\cdot\lim_{\theta\to\frac{\pi}{4}} e^{\log\left((\tan\theta)^{\tan 2\theta}\right)} =$

fcn e^y is cts $\Rightarrow$ $= 3\cdot e^{\lim_{\theta\to\pi/4}\tan(2\theta)\cdot\log\tan\theta} = 3\cdot e^{-1} = \frac{3}{e}.$

Problem: Is $f(x) = \begin{cases} \sqrt{1+x^2}-1 & x \le 0 \\ x^2\cos\left(\frac{1}{x}\right) & x > 0 \end{cases}$ diff at $x=0$?

(final, 2015)

Solution: $f'(0)$ exists if $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists

Now $f(0) = \sqrt{1+0^2} - 1 = 0$ ← just from def'n of f.

so we need to evaluate $\lim\limits_{x\to 0} \frac{f(x)}{x}$.

~~Lees~~ On the left, this reads

$$\lim_{x\to 0^-} \frac{\sqrt{1+x^2}-1}{x} = \lim_{x\to 0^-} \frac{\sqrt{1+x^2}-1}{x} \cdot \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} = \lim_{x\to 0^-} \frac{(1+x^2)-1}{x\left(1+\sqrt{1+x^2}\right)} =$$

$$= \lim_{x\to 0^-} \frac{x}{1+\sqrt{1+x^2}} = \frac{0}{1+\sqrt{1+0^2}} = 0$$

On the right, the limit is $\lim\limits_{x\to 0^+} \frac{x^2\cos\left(\frac{1}{x}\right)}{x} = \lim\limits_{x\to 0^+} x\cos\left(\frac{1}{x}\right)$

(note: as $x\to 0$, $x \to 0$ but $\cos\left(\frac{1}{x}\right)$ is bounded, doesn't pull to ∞)

for any $x \neq 0$, $1 \le \cos\left(\frac{1}{x}\right) \le -1$, so for $x>0$, $-x \le x\cos\left(\frac{1}{x}\right) \le x$

and $\lim\limits_{x\to 0} x = \lim\limits_{x\to 0}(-x) = 0$, so by the squeeze thm, $\lim\limits_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = 0$.

Endgame: We have seen $\lim\limits_{x\to 0^-} \frac{f(x)}{x} = \lim\limits_{x\to 0^+} \frac{f(x)}{x} = 0$, so $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$

and $f'(0)$ exists (and equals 0)