Math 100 – SOLUTIONS TO WORKSHEET 2
LIMIT LAWS

1. Existence of limits and blowup

(1) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

Solution: $f(x) = \frac{x-3}{(x-3)(x-2)} = \frac{1}{x-2}$ so $\lim_{x\to 3} f(x) = \frac{1}{3-2} = \boxed{1}$.

(b) What about $\lim_{x\to 2} f(x)$? What about $\lim_{x\to -2^+} f(x)$, $\lim_{x\to 2^-} f(x)$?

Solution: The limits do not exist: if x is very close to 2 then $x-2$ is very small and $\frac{1}{x-2}$ is very large. That said, when $x > 2$ we have $\frac{1}{x-2} > 0$ and when $x < 2$ we have $\frac{1}{x-2} < 0$ so (in the extended sense)

$$\lim_{x\to 2^+} \frac{1}{x-2} = +\infty$$
$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty\,.$$

(c) (Final, 2014) Evaluate $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

Solution: The denominator vanishes at -3 while the numerator does not, so the function blows up there. When $x > -3$, we have $x+3 > 0$. Also, when x is close to -3, $x+2$ is close to -1. We conclude that $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$.

(2) Evaluate

(a) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

Solution: The function blows up at both sides, and remains positive on both sides. Therefore

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty\,.$$

(b) $\lim_{x\to\pi+} \frac{1}{\sin(x)}$, $\lim_{x\to\pi-} \frac{1}{\sin(x)}$.

Solution: $\lim_{x\to\pi} \sin(x) = 0$ so the function blows up there. When $x < \pi$ the function is positive, while for $x > \pi$ the function is negative. We therefore have

$$\lim_{x\to\pi^+} \frac{1}{\sin x} = -\infty$$
$$\lim_{x\to\pi^-} \frac{1}{\sin x} = +\infty$$

(3) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2-x^2) = 2 - 1^2 = 1$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$ so

$$\lim_{x\to 1} f(x) = 1\,.$$

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

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Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4-x^2) = 4-1^2 = 3$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\sqrt{x} = \sqrt{1} = 1$ so the limit does not exist (but the one-sided limits do).

2. Limit Laws

Fact. *Limits respect arithmetic operations and standard functions (e^x, sin, cos, log, ...)* **as long as everything is well-defined.**

(beware especially of division by zero)

(4) Evaluate using the limit laws:

(a) $\lim_{x\to 2}\frac{x+1}{4x^2-1} =$

Solution: The expression is well-behaved at $x=2$ so $\lim_{x\to 2}\frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2-1} = \frac{3}{15} = \frac{1}{5}$.

(b) $\lim_{x\to 1}\frac{e^x(x-1)}{x^2+x-2} =$

Solution: $\lim_{x\to 1}\frac{e^x(x-1)}{x^2+x-2} = \lim_{x\to 1}\frac{e^x(x-1)}{(x-1)(x+2)} = \lim_{x\to 1}\frac{e^x}{x+2} = \frac{e^1}{1+2} = \frac{e}{3}$.

(5) Evaluate:

(a) $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}$.

Solution: Both numerator and denominator vanish at $x=0$ so we need to deal with the cancellation. Multiplying and dividing by $\sqrt{4+x}+2$ we have

$$\begin{aligned}\frac{\sqrt{4+x}-2}{x} &= \frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}\\ &= \frac{(4+x)-4}{x\left(\sqrt{4+x}+2\right)} = \frac{x}{x\left(\sqrt{4+x}+2\right)}\\ \frac{1}{\sqrt{4+x}+2} &\xrightarrow[x\to 0]{} \frac{1}{\sqrt{4}+2} = \frac{1}{4}\,.\end{aligned}$$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$.

Solution: We have

$$\begin{aligned}\frac{\sqrt{1+x}-\sqrt{1+x^2}}{x^2} &= \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x^2}\cdot\frac{\sqrt{1+x}+\sqrt{1+x^2}}{\sqrt{1+x}+\sqrt{1+x^2}}\\ &= \frac{(1+x)-(1+x^2)}{x^2\left(\sqrt{1+x}+\sqrt{1+x^2}\right)}\\ &= \frac{x-x^2}{x^2\left(\sqrt{1+x}+\sqrt{1+x^2}\right)}\\ &= \frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}}\cdot\frac{1}{x}\,.\end{aligned}$$

Now as $x\to 0$ we have $\frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}} \to \frac{1}{2}$ while $\frac{1}{x}$ blows up so the whole expression blows up and the limit does not exist.

(c) $\lim_{x\to 0} x^2\sin\left(\frac{\pi}{x}\right)$.

Solution: Since $-1 \le \sin\theta \le 1$ for all θ while $x^2 \ge 0$ we have for all x that

$$-x^2 \le x^2\sin\left(\frac{\pi}{x}\right) \le x^2\,.$$

Now $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}(-x^2) = 0$, so by the sandwich theorem $\lim_{x\to 0} x^2\sin\left(\frac{\pi}{x}\right) = 0$ too.

(d) (Final, 2014) Suppose that $8x \le f(x) \le x^2+16$ for all $x\ge 0$. Find $\lim_{x\to 4} f(x)$.

Solution: We have $\lim_{x\to 4} 8x = 32$ and $\lim_{x\to 4} x^2+16 = 32$ so by the sandwich theorem $\lim_{x\to 4} f(x)$ exists and equals 32.