Math 100 – SOLUTIONS TO WORKSHEET 9
LOGARITHMS AND LOGARITHMIC DIFFERENTIATION

1. Review of Logarithms

(1) $\log\left(e^{10}\right) =$ $\log(2^{100}) =$
Solution: $\log e^{10} = 10$ while $\log(2^{100}) = 100\log 2$.

(2) A variant on *Moore's Law* states that computing power doubles every 18 months. Suppose computers today can do N_0 operations per second.
(a) Write a formula predicting the future:

- Computers t years from now will be able to do $N(t)$ operations per second where

$$N(t) =$$

Solution: Since there is a doubling every 18 months, there will be $t/1.5$ doublings in t years and $N(t) = N_0 2^{t/1.5}$.

(b) A computing task would take 10 years for today's computers. Suppose we wait 3 years and then start the computation. When will we have the answer?
Solution: There will be two doublings in 3 years, so we will have the answer $3 + \frac{10}{2^2} = 3 + \frac{10}{4} =$ 5.5 years from now.

(c) At what time will computers be powerful enough to complete the task in 6 months?
Solution: The computers of t years from now will be able to complete the task in $10 \cdot 2^{-t/1.5}$ years, so we need to find t such that

$$10 \cdot 2^{-t/1.5} = \frac{1}{2}.$$

This is equivalent to $2^{t/1.5} = 20$, and taking logarithms gives

$$\frac{t}{1.5}\log 2 = \log 20$$

and hence

$$\boxed{t = 1.5\frac{\log 20}{\log 2}}.$$

Solution: Can also write $2^{-t/1.5} = \frac{1}{20}$ and take logarithms to get $-\frac{t}{1.5} = \frac{\log\frac{1}{20}}{\log 2}$ so that $t = -1.5\frac{\log\frac{1}{20}}{\log 2} = 1.5\frac{\log 20}{\log 2}$ since $\log\frac{1}{20} = -\log 20$.

2. Differentiation

$$\boxed{(\log x)' = \frac{1}{x}}$$

(1) Differentiate
(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2 + 3t\right) =$
Solution: By the chain rule, the derivatives are: $\frac{1}{ax} \cdot a = \frac{1}{x}$ and $\frac{1}{t^2+3t} \cdot (2t+3) = \frac{2t+t}{t^2+3t}$. We can also use the logarithm laws first: $\log(ax) = \log a + \log x$ so $\frac{\mathrm{d}}{\mathrm{d}x}(\log ax) = \frac{\mathrm{d}}{\mathrm{d}x}(\log a) + \frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}$ since $\log a$ is constant if a is. Similarly, $\log(t^2+3t) = \log t + \log(t+3)$ so its derivative is $\frac{1}{t} + \frac{1}{t+3}$.

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(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1+x^2) =$ $\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$

Solution: Applying the product rule and then the chain rule we get: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\log(1+x^2)\right) = 2x\log(1+x^2) + x^2\frac{1}{1+x^2}\cdot 2x = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$. Using the quotient rule and the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} = -\frac{1}{\log^2(2+\sin r)}\cdot\frac{1}{2+\sin r}\cdot\cos r = -\frac{\cos r}{(2+\sin r)\log^2(2+\sin r)}\,.$$

(2) (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}$.

Solution: We have

$$\begin{aligned}\log y &= \log\left(x^2+1\right) + \log(\sin x) + \log\left(\frac{1}{\sqrt{x^2+3}}\right) + \log\left(e^{\cos x}\right)\\ &= \log\left(x^2+1\right) + \log\left(\sin x\right) - \frac{1}{2}\log\left(x^2+3\right) + \cos x\,.\end{aligned}$$

Differentiating with respect to x gives:

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{2x}{x^2+3} - \sin x$$

and solving for y' finally gives

$$y' = \left(\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{x}{x^2+3} - \sin x\right)\cdot\left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}\,.$$

(3) Differentiate using $\boxed{f' = f\times(\log f)'}$

(a) x^x

Solution: If $y = x^x$ then $\log y = x\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \log x + x\cdot\frac{1}{x} = \log x + 1$ so $y' = y\left(\log x + 1\right) = x^x\left(\log x + 1\right)$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right) = x^x\frac{\mathrm{d}}{\mathrm{d}x}\left(\log(x^x)\right) = x^x\left(\log x + 1\right)$.

Solution: We have $x^x = \left(e^{\log x}\right)^x = e^{x\log x}$. Applying the chain rule we now get $\left(x^x\right)' = e^{x\log x}\left(\log x + 1\right) = x^x\left(\log x + 1\right)$.

(b) $(\log x)^{\cos x}$

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x\right)^{\cos x} &= \left(\log x\right)^{\cos x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\log(\log x)\right)\\ &= -\sin x\log\log x\left(\log x\right)^{\cos x} + \left(\log x\right)^{\cos x}\cos x\frac{1}{\log x}\frac{1}{x}\\ &= -\sin x\log\log x\left(\log x\right)^{\cos x} + \cos x\left(\log x\right)^{\cos x - 1}\frac{1}{x}\,.\end{aligned}$$

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(c) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = y\frac{\mathrm{d}\log y}{\mathrm{d}x} &= x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x\cdot\log x\right)\\ &= x^x\left(2\log x\cdot\frac{1}{x}\right) = 2\log x\cdot x^{x-1}\,.\end{aligned}$$