Math 100 – SOLUTIONS TO WORKSHEET 11
SCIENTIFIC APPLICATIONS

1. Velocity and acceleration

(1) A particle's position is given by $f(t) = \frac{1}{\pi}\sin(\pi t)$.

(a) Find the velocity at time t, and specifically at $t = 3$.
Solution: The velocity is the derivative of the position, so $v(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = \cos(\pi t)$. In particular $v(3) = \cos(3\pi) = -1$.

(b) When is the particle moving to the right? to the left?
Solution: The particular is moving to the right when $\cos(\pi t) > 0$ i.e. for times $t \in \left(-\frac{1}{2}, \frac{1}{2}\right) + 2\mathbb{Z}$. It is moving to the left for $t \in \left(\frac{1}{2}, \frac{3}{2}\right) + 2\mathbb{Z}$.

(c) When is the particle accelerating? decelerating?
Solution: The acceleration is $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\pi\sin(\pi t)$. This is positive when $t \in (1,2) + 2\mathbb{Z}$, negative when $t \in (0,1) + 2\mathbb{Z}$. But "accelerating" means the acceleration is in the same direction as the velocity! So the particle is accelerating for $t \in \left(\left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)\right) + 2\mathbb{Z}$ and decelerating for $t \in \left(\left(0, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)\right) + 2\mathbb{Z}$.

(2) (Final, 2016) An object is thrown straight up into the air at time $t = 0$ seconds. Its height in metres at time t seconds is given by $h(t) = s_0 + v_0 t - 5t^2$. In the first second the object rises by 5 metres. For how many seconds does the object rise before beginning to fall?

Solution: We are given that $h(1) - h(0) = 5$, in other words that $(s_0 + v_0 - 5) - s_0 = 5$ so that $v_0 = 10$. Now the velocity of the object is

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = v_0 - 10t = 10 - 10t$$

and this is positive as long as $t \leq 1$s.

(3) A emergency breaking car can decelerate at $9\frac{\mathrm{m}}{\mathrm{s}^2}$. How fast can a car drive so that it can come to a stop within 50m?

Solution: Suppose the car beings with velocity v_0. Its velocity at time t is then $v(t) = v_0 - gt$ so the stopping time is $t = \frac{v_0}{g}$. Reversing time, the distance travelled during the deceleration is the same as the distance travelled while accelerating at acceleration g for time t. The breaking distance L therefore has the form $\frac{1}{2}gt^2 = \frac{v_0^2}{2g}$. The maximum speed is then

$$v_0 = \sqrt{2gL} = \sqrt{2 \cdot 9 \cdot 50} = 30\frac{\mathrm{m}}{\mathrm{s}} = 180\mathrm{km/h}.$$

2. Other applications

(1)

(a) Water is filling a cylindrical container of radius $r = 10$cm. Suppose that at time t seconds the height of the water is $\left(t + t^2\right)$ cm. How fast is the volume growing?
Solution: At every time the water fills a cylinderical volume of radius r height $h(t) = (t + t^2)$ hence volume

$$V(t) = \pi r^2 h(t) = 100\pi(t + t^2)\mathrm{cm}^3 .$$

We therefore have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 100\pi(1 + 2t)\frac{\mathrm{l}}{\mathrm{s}} .$$

Date: 10/10/2019, Worksheet by Lior Silberman.

(b) A rocket is flying in space. The momentum of the rocket is given by the formula $p = mv$, where m is the mass and v is the velocity. At a time where the mass of the rocket is $m = 1000\text{kg}$ and its velocity is $v = 500\frac{\text{m}}{\text{s}}$ the rocket is accelerating at the rate $a = 20\frac{\text{m}}{\text{s}^2}$ and losing mass at the rate $10\frac{\text{kg}}{\text{s}}$. Find the rate of change of the momentum with time.

Solution: By the product rule we have

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}m}{\mathrm{d}t} \cdot v + m\frac{\mathrm{d}v}{\mathrm{d}t} \\ &= (-10 \cdot 500 + 1000 \cdot 20)\,\frac{\text{kgm}}{\text{s}^2}\end{aligned}$$

(2) A ball is falling from rest in air. Its height at time t is given by

$$h(t) = H_0 - gt_0\left(t + t_0 e^{-t/t_0} - t_0\right)$$

where H_0 is the initial height and t_0 is a constant.

(a) Find the velocity of the ball. $v(t) =$

Solution: We have $v(t) = \frac{\mathrm{d}h}{\mathrm{d}t} = -gt_0\left(1 - e^{-t/t_0}\right)$.

(b) Find the acceleration. $a(t) =$

Solution: We have $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -ge^{-t/t_0}$.

(c) Find $\lim_{t\to\infty} v(t)$

Solution: The limit is $\lim_{t\to\infty} v(t) = -gt_0 \lim_{t\to\infty}\left(1 - e^{-t/t_0}\right) = -gt_0$.