Math 100 – SOLUTIONS TO WORKSHEET 18
THE MVT AND CURVE SKETCHING

1. Applying the MVT

(1) Suppose $f'(x) = \frac{e^x}{x+\pi}$ for $0 \le x \le 2$. Give an upper bound for $f(2) - f(0)$.
Solution: We are given that f is differentiable on the interval, so by the MVT there is $c \in (0,2)$ such that $\frac{f(2)-f(0)}{2-0} = f'(c)$ and hence $f(2) - f(0) = 2f'(c)$. Now $f'(c) = \frac{e^c}{c+\pi}$. Since $c \le 2$ we have $e^c \le e^2$ and since $c \ge 0$ we have $\frac{1}{c+\pi} \le \frac{1}{\pi}$ so $f'(c) \le \frac{e^2}{\pi}$ and $f(2) - f(0) \le \frac{2e^2}{\pi}$.

(2) (Final, 2015) Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at most two solutions.
Solution: See WS 17

(3) Suppose f satisfies the hypotheses of the MVT and that $f'(x) > 0$ for all $x \in (a,b)$. Show that $\frac{f(b)-f(a)}{b-a} > 0$, and hence that $f(b) > f(a)$.
Solution: There is $c \in (a,b)$ so that
$$\frac{f(b)-f(a)}{b-a} = f'(c) > 0\,.$$
Multiplying by the positive quantity $b - a$ we find $f(b) - f(a) > 0$, that is
$$f(b) > f(a)\,.$$

2. The shape of a the graph

(4) Let f be twice differentiable on $[a,b]$.
(a) Suppose first that $f(a) = f(b) = 0$ and that f is positive somewhere between a, b. Show that there is c between a, b so that $f''(c) < 0$.
Solution: Suppose $a < x < b$ so that $f(x) > 0$. By the MVT there are $y \in (a,x)$ and $z \in (x,b)$ so that
$$f'(y) = \frac{f(x)-f(a)}{x-a} = \frac{f(x)}{x-a} > 0$$
$$f'(z) = \frac{f(b)-f(x)}{b-x} = -\frac{f(x)}{b-a} < 0\,.$$
In particular, $f'(y) > f'(z)$ but $y < z$ so $f'(z) - f'(y) < 0$ but $z - y > 0$. Applying the MVT to the twice differentiable function f' on the interval $[y,z]$ gives $c \in [y,z] \subset (a,b)$ such that
$$f''(c) = \frac{f'(z) - f'(y)}{z-y} < 0\,.$$
(b) Now let $f(a), f(b)$ take any values, but suppose $f''(x) > 0$ on (a,b). Let $L : y = mx + n$ be the line through $(a, f(a)), (b, f(b))$. Applying part (a) to $g(x) = f(x) - (mx+n)$ show that the graph of f lies below the line L.
Solution: Let $g(x) = f(x) - (mx+n)$. Since the line passes through $(a, f(a))$ and $(b, f(b))$ we have $g(a) = g(b) = 0$. Also, for all $a < x < b$, $g''(x) = f''(x) > 0$ since $(mx+n)'' = 0$. It follows that there is no point such that $g(x) > 0$, so $g(x) \le 0$ that is $f(x) \le mx + n$.

Definition. We say f is *concave up* (or "convex") on an interval $[a,b]$ if its graph lies under the secant lines in this interval. This is true, for example, if $f'' > 0$ on (a,b). We say f is *concave down* (or "concave") on the interval if its graph lies below the scant lines, in particular when $f'' < 0$ on (a,b). We say that f has an *inflection point* at x_0 if its second derivative changes sign there.

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