Lior Silberman's Math 412: Problem Set 6 (due 22/10/2019)

P1. (Minimal polynomials)

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$.

(a) Find the minimal polynomial of A and show that the minimal polynomial of B is $x^2(x-1)^2$.

(b) Find a 3×3 matrix whose minimal polynomial is x^2.

P2. For each of A, B find its eigenvalues and the correpsonding generalized eigenspaces.

Triangular matrices

P3. Let L be a lower-triangular square matrix with non-zero diagonal entries. Find a formula for its inverse.

1. Let U be an upper-triangular square matrix with non-zero diagonal entries.

(a) Give a "backward-substitution" algorithm for solving $U\underline{x} = \underline{b}$ efficiently.

(b) Explicitely use your algorithm to solve $\begin{pmatrix} 1 & 4 & 5 \\ & 2 & 6 \\ & & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

(c) For a general upper-triangular U give a formula for U^{-1}, proving in particular that U is invertible and that U^{-1} is again upper-triangular.

RMK We'll see that if $\mathcal{A} \subset M_n(F)$ is a subspace containing the identity matrix and closed under matrix multiplication, then the inverse of any matrix in $\mathcal{A}$ belongs to $\mathcal{A}$. This applies, in particular, to the set of upper-triangular matrices.

The minimal polynomial

2. Let $D \in M_n(F) = \operatorname{diag}(a_1, \ldots, a_n)$ be diagonal.

(a) For any polynomial $p \in F[x]$ show that $p(D) = \operatorname{diag}(p(a_1), \ldots, p(a_n))$.

(b) Show that the minimal polynomial of D is $m_D(x) = \prod_{j=1}^{r}(x - a_{i_j})$ where $\left\{a_{i_j}\right\}_{j=1}^{r}$ is an enumeration of the distinct values among the a_i.

(c) Show that (over any field) the matrix B from problem P1 is not similar to a diagonal matrix.

(d) Now suppose that U is an upper-triangular matrix with diagonal D. Show that for any $p \in F[x]$, $p(U)$ has diagonal $p(D)$. In particular, $m_D | m_U$.

3. Let $T \in \operatorname{End}(V)$ be diagonable. Show that every generalized eigenspace is simply an eigenspace.

4. Let $S \in \operatorname{End}(U)$, $T \in \operatorname{End}(V)$. Let $S \oplus T \in \operatorname{End}(U \oplus V)$ be the "block-diagonal map".

(a) For $f \in F[x]$ show that $f(S \oplus T) = f(S) \oplus f(T)$.

(b) Show that $m_{S \oplus T} = \operatorname{lcm}(m_S, m_T)$ ("least common multiple": the polynomial of smallest degree which is a multiple of both).

(c) Conclude that $\operatorname{Spec}_F(S \oplus T) = \operatorname{Spec}_F(S) \cup \operatorname{Spec}_F(T)$.

RMK See also problem B below.

5. Let K/F be an extension of fields, let V be a finite-dimensional F-vectorpsace and let $T \in \mathrm{End}_F(V)$. Show that the minimal and charcteristic polynomials of $T_K \in \mathrm{End}_K(V_K)$ are identical with those of T.

Extra credit

6. Let $R \in \mathrm{End}(U \oplus V)$ be "block-upper-triangular", in that $R(U) \subset U$.
   (a) Define a "quotient linear map" $\bar{R} \in \mathrm{End}(U \oplus V/U)$.
   (b) Let S be the restriction of R to U. Show that both m_S, $m_{\bar{R}}$ divide m_R.
   (c) Let $f = \mathrm{lcm}[m_S, m_{\bar{R}}]$ and set $T = f(R)$. Show that $T(U) = \{\underline{0}\}$ and that $T(V) \subset U$.
   (d) Show that $T^2 = 0$ and conclude that $f \mid m_R \mid f^2$.
   (e) Show that $\mathrm{Spec}_F(R) = \mathrm{Spec}_F(S) \cup \mathrm{Spec}_F(\bar{R})$.

Supplementary problems

A. (Cholesky decomposition)
   (a) Let A be a positive-definite square matrix. Show that $A = LL^\dagger$ for a unique lower-triangular matrix L with positive entries on the diagonal.

   DEF For $\varepsilon \in \pm 1$ define $D^\varepsilon \in M_n(\mathbb{R})$ by $D^\varepsilon_{ij} = \begin{cases} \varepsilon & j = i + \varepsilon \\ -\varepsilon & j = i \\ 0 & j \neq i, i+\varepsilon \end{cases}$ and let $A = -D^- D^+$ be the (positive) discrete Laplace operator.

   (b) To $f \in C^\infty(0,1)$ associate the vector $\underline{f} \in \mathbb{R}^n$ where $\underline{f}(i) = f(\frac{i}{n})$. Show that $\frac{1}{n}D^+\underline{f}$ and $\frac{1}{n}D^-\underline{f}$ are both close to $\underline{f'}$ (so that both are discrete differentiation operators). Show that $\frac{1}{n^2}D^-D^+$ is an approximation to the second derivative.
   (c) Find a lower-triangular matrix L such that $LL^\dagger = A$.

B. Let $T \in \mathrm{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \left\{\underline{v} \in V \mid \exists k : p(T)^k \underline{v} = \underline{0}\right\}$.
   (a) Show that V_p is a T-invariant subspace of V and that $m_{T\restriction V_p} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{\underline{0}\}$. Conclude that $p^k | m_T$.
   (b) Show that if $\{p_i\}_{i=1}^r \subset F[x]$ are distinct monic irreducibles then the sum $\bigoplus_{i=1}^r V_{p_i}$ is direct.
   (c) Let $\{p_i\}_{i=1}^r \subset F[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^r V_{p_i}$.
   (d) Suppose that $m_T(x) = \prod_{i=1}^r p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \mathrm{Ker}\, p_i^{k_i}(T)$.

Lior Silberman's Math 412: Solutions to Problem Set 6

P1. (Minimal polynomials)

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ does not satisfy any linear polynomial (if $aA + b\,\mathrm{Id} = 0$ then $A = -\frac{b}{a}\,\mathrm{Id}$ would be scalar. However, $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} + \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} = 5A + 2I$ so $A^2 - 5A + 2I = 0$ and this is the minimal polynomial.

$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$ has $B^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & 1 \end{pmatrix}$ and $(B-1)^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. It is then easy to check that $B^2(B-1)^2 = 0$. Thus the minimal polynomial must be a divisor of $x^2(x-1)^2$, and by the unique factorization theorem for polynomials any such divisor divides one of $x^2(x-1)$ and $x(x-1)^2$. However, $B^2(B-1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & * & * & * \\ * & * & * & * \end{pmatrix} \neq 0$ and similarly $B(B-1)^2 \neq 0$.

(b) Let $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $N^2 = 0$ so the minimal polynomial is a divisor of x^2. The only proper divisor is x, and isn't the mininal polynomial since $N \neq 0$.

P2. The eigenvalues are the roots of the minimal polynomial. For A these are $\frac{5\pm\sqrt{17}}{2}$. For B these are $0,1$. The generalized eigenspaces for A are simply the eigenspaces spanned by the eigenvectors. The rest of the discussion focuses on B.
Let $U_0 = \mathrm{Ker}\,B^2$, $U_1 = \mathrm{Ker}\,(B-1)^2$. Adding $\frac{3}{2}$ the last row to the third (assume 2 is invertible) we see that $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/2 \\ 2 & 2 & -2 & 1 \end{pmatrix}$. It follows that $U_0 = \{(x,y,x+y,0)^t\}$ (in characteristic 2, $B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and we get the same conclusion). Similarly, $U_1 = \{(0,0,z,w)^t\}$. Let V_0, V_1 be the generalized eigenspaces. Then certainly $U_0 \subset V_0$ and $U_1 \subset V_1$. Also, U_0, U_1 are each 2-dimensional and their intersection is

empty. It follows that the sum U_0+U_1 is direct and 4-dimensional, that is $F^4=U_0\oplus U_1$. This means that $V_0=U_0$ and $V_1=U_1$: let $\underline{v}\in V_0$, for example. Then $\underline{v}=\underline{u}_0+\underline{u}_1$ for $\underline{u}_\lambda\in U_\lambda$. Apply B^k for $k\geq 2$ such that $B\underline{v}=\underline{0}$. Then also $B^k\underline{u}_0=\underline{0}$ and so $B^k\underline{u}_1=\underline{0}$. This contradicts B being invertible on V_1 unless $\underline{u}_1=\underline{0}$ so that $\underline{v}=\underline{u}_0\in U_0$. Similarly, if $\underline{v}\in V_1$ then applying $(B-I)^k$ to $\underline{v}$ shows that $\underline{v}\in U_1$.

2. (a) We have $D^0=\mathrm{Id}=\mathrm{diag}\,(1,\ldots,1)=\mathrm{diag}\left(a_1^0,\ldots,a_n^0\right)$. Suppose that for some $k\geq 0$ we have $D^k=\mathrm{diag}\left(a_1^k\ldots,a_n^k\right)$. Then $D^{k+1}=DD^k=\mathrm{diag}\,(a_i)\,\mathrm{diag}\left(a_i^k\right)=\mathrm{diag}\left(a_i^{k+1}\right)$. Finally, let $p(x)=\sum_{k=0}^K\alpha_kx^k$. Then

$$p(D)=\sum_{k=0}^K\alpha_kD^k=\sum_{k=0}^K\alpha_k\,\mathrm{diag}\left(a_i^k\right)=\sum_{k=0}^K\mathrm{diag}\left(\alpha_ka_i^k\right)=\mathrm{diag}\left(\sum_{k=0}^K\alpha_ka_i^k\right)=\mathrm{diag}\,(p(a_i))\,.$$

   (b) Let $p_D(x)$ be the given polynomial. Then for each a_i we have $p_D(a_i)=0$ and hence $p_D(D)=\mathrm{diag}\,(0)=0$, so the minimal polynomial divides p_D.On the other hand, each a_i is an eigenvalue of D, hence a zero of m_D. It follows that $m_D=p_D$.
   (c) Its minimal polynomial has multiple roots.
   (d) Let U,U' be upper-triangular. We then have $(\alpha U+U')_{ii}=\alpha U_{ii}+U'_{ii}$ and, since $U_{ij}=0$ if $j<i$ and $U'_{ji}=0$ if $j>i$ we have

$$\left(UU'\right)_{ii}=\sum_jU_{ij}U'_{ji}=\sum_{i\leq j\leq i}U_{ij}U'_{ji}=U_{ii}U'_{ii}\,.$$

   Now the same induction argument as in (a) shows that $(p(U))_{ii}=p\left(U_{ii}\right)$. In particular, if $p(U)=0$ then $p(D)=0$ and so $m_D|m_U$.

3. Let $U_\lambda\subset V$ be the eigenspaces of T, V_λ the generalized eigenspaces.
   (1) Fix an eigenbasis $B\subset V$. Now suppose that $(T-\lambda)^k\underline{v}=\underline{0}$ for some $\underline{v}\in V$. We have $\underline{v}=\sum_{i=1}^na_i\underline{v}_i$ for some $a_i\in F$ and $\underline{v}_i\in B$. Suppose $T\underline{v}_i=\lambda_i\underline{v}_i$. Then

$$\underline{0}=(T-\lambda)^k\underline{v}=\sum_{i=1}^na_i\left(\lambda_i-\lambda\right)^k\underline{v}_i\,.$$

   Since B is a basis it follows that $a_i\left(\lambda_i-\lambda\right)^k=0$ for each i, and if $\lambda_i\neq\lambda$ this means $a_i=0$. It follows that

$$\underline{v}=\sum_{\lambda_i=\lambda}a_i\underline{v}_i\in U_\lambda\,.$$

   (2) We have $U_\lambda\subset V_\lambda$ and at the same time $V=\bigoplus_{\lambda\in\mathrm{Spec}_F(T)}U_\lambda$ (by assumption) and $V=\bigoplus_{\lambda\in\mathrm{Spec}_F(T)}V_\lambda$ (theorem from class). If V is finite-dimensional then we have

$$\dim_FV=\sum_\lambda\dim_FU_\lambda\leq\sum_\lambda\dim_FV_\lambda=\dim V$$

   and we must therefore have equality throughout, that is $\dim_FU_\lambda=\dim_FV_\lambda$ and $U_\lambda=V_\lambda$.
   (3) For each λ let $\iota_\lambda\colon V_\lambda\to\bigoplus_\lambda V_\lambda$ be the standard map. Let $W\subset\bigoplus_\lambda V_\lambda$ be the internal direct sum of the images of the U_λ. Composing with the quotient map $\bigoplus_\lambda V_\lambda\to\bigoplus_\lambda V_\lambda/W$ gives a map

$$f_\lambda\colon V_\lambda\to\bigoplus_\lambda V_\lambda\Big/\bigoplus_\lambda U_\lambda\,.$$

Note that if $\underline{v} \in U_\lambda$ then $\iota_\lambda(\underline{v})$ is in W, and so $f_\lambda(\underline{v}) = \underline{0}$. It follows that f_λ induces a map

$$\bar{f}_\lambda : V_\lambda / U_\lambda \to \bigoplus_\lambda V_\lambda \Big/ \bigoplus_\lambda U_\lambda .$$

Finally, this family of maps induces a map

$$\bar{f} \colon \bigoplus_\lambda (V_\lambda / U_\lambda) \to \bigoplus_\lambda V_\lambda \Big/ \bigoplus_\lambda U_\lambda .$$

This is an isomorphism: if $\underline{v} \in \bigoplus_\lambda \underline{v}_\lambda$ then $\underline{v} = \sum_i \underline{v}_i$ for some $\underline{v}_i \in V_{\lambda_i}$ and then $\bar{f}\left(\sum_i \left(\underline{v}_i + U_{\lambda_i}\right)\right) = \underline{v} + W$, and if $\bar{f}\left(\sum_i \left(\underline{v}_i + U_{\lambda_i}\right)\right) = 0$ (λ_i distinct) then $\sum_i \underline{v}_i \in W$ so each $\underline{v}_i \in U_{\lambda_i}$. Now we are given that $\bigoplus_\lambda V_\lambda \big/ \bigoplus_\lambda U_\lambda$ is the zero space (both spaces are isomorphic to V) so $\bigoplus_\lambda (V_\lambda / U_\lambda)$ is zero, and hence for each λ $V_\lambda / U_\lambda = \{\underline{0}\}$ and $V_\lambda = U_\lambda$.

4. Let $S \in \mathrm{End}(U)$, $T \in \mathrm{End}(V)$. Let $S \oplus T \in \mathrm{End}(U \oplus V)$ be the "block-diagonal map".
   (a) Let $S_1, S_2 \in \mathrm{End}(U)$, $T_1, T_2 \in \mathrm{End}(V)$ and let $\alpha \in F$. Then

$$\begin{aligned}
[\alpha(S_1 \oplus T_1) + (S_2 \oplus T_2)](\underline{u} \oplus \underline{v}) &= \alpha(S_1 \oplus T_1)(\underline{u} \oplus \underline{v}) + (S_2 \oplus T_2)(\underline{u} \oplus \underline{v}) \\
&= \alpha(S_1\underline{u} \oplus T_1\underline{v}) + (S_2\underline{u} \oplus T_2\underline{v}) \\
&= (\alpha S_1 + S_2)\underline{u} \oplus (\alpha T_1 + T_2)\underline{v} \\
&= [(\alpha S_1 + S_2) \oplus (\alpha T_1 + T_2)](\underline{u} \oplus \underline{v})
\end{aligned}$$

   and

$$\begin{aligned}
[(S_1 \oplus T_1)(S_2 \oplus T_2)](\underline{u} \oplus \underline{v}) &= (S_1 \oplus T_1)(S_2\underline{u} \oplus T_2\underline{v}) \\
&= (S_1S_2\underline{u}) \oplus (T_1T_2\underline{v}) \\
&= (S_1S_2 \oplus T_1T_2)(\underline{u} \oplus \underline{v}) .
\end{aligned}$$

   Now from the second claim it follows by induction on k that $(S \oplus T)^k = S^k \oplus T^k$, and then it follows by induction on n that

$$\sum_{k=0}^n \alpha_k (S \oplus T)^k = \sum_{k=0}^n \alpha_k \left(S^k \oplus T^k\right) = \left(\sum_{k=0}^n \alpha_k T^k\right) \oplus \left(\sum_{k=0}^n \alpha_k S^k\right) .$$

   (b) Let $f = \mathrm{lcm}(m_S, m_T)$. Then $f(S) = 0$ and $f(T) = 0$ (f is a multiple of the respective minimal polynomials), and hence $f(S \oplus T) = f(S) \oplus f(T) = 0 \oplus 0$, so f is divisible by the minimal polynomial of $S \oplus T$. Conversely, we have $m_{S\oplus T}(S \oplus T) = m_{S\oplus T}(S) \oplus m_{S\oplus T}(T) = 0$ so $m_{S\oplus T}(S) = 0$ and $m_{S\oplus T}(T) = 0$. It follows that $m_{S\oplus T}$ is divisible by both m_S and m_T, hence by their least common multiple.
   (c) Clearly if $m_S(\lambda) = 0$ or $m_T(\lambda) = 0$ then $f(\lambda) = 0$ (it's a multiple). For the converse, let λ be a root of f, but not of m_S or m_T. Then $x - \lambda$ divides f by not m_S or m_T, so both m_S and m_T divide $\frac{f(x)}{x-\lambda}$, contradicting the minimality of f.
5. Let $R \in \mathrm{End}(U \oplus V)$ be "block-upper-triangular", in that $R(U) \subset U$.
   (a) In general, if $T \in \mathrm{End}(W)$ and $Z \subset W$ is T-stable then setting $\bar{T}(\underline{w} + Z) = T\underline{w} + Z$ gives a linear map.
   (b) For any polynomial f and any $\underline{u} \in U$ we have $f(R)\underline{u} = f(S)\underline{u}$, by the same induction as the in the problems above. In particular, if $f(R) = 0$ then $f(S) = 0$ and $m_S | f$. Similarly, for $\underline{w} \in U \oplus V$, $f(R)(\underline{w} + U) = f(R)\underline{w} + U$ so that $f(\bar{R}) = \overline{f(R)}$. In particular, if $f(R) = 0$ then $f(\bar{R}) = 0$ and $m_{\bar{R}} | f$.

(c) Let $f = \mathrm{lcm}[m_S, m_{\bar{R}}]$ and set $T = f(R)$. Since $m_S | f$ we have $f(S) = 0$. Then for $\underline{u} \in U$ we have $T\underline{u} = f(S)\underline{u} = 0$, so $T(U) = 0$. For the same reason, $f(\bar{R}) = 0$, that is $\bar{T} = 0$ which means $T(V) \subset U$.

(d) Let $T(V) \subset U$ and $T(U) = 0$ we have $T^2 = 0$, so $f^2(R) = 0$ and hence $m_R | f^2$. We have already seen that $f | m_R$.

(e) Since $f | m_R | f^2$, any root of f is a root of m_R and any root of m_R is a root of f^2. But f, f^2 have the same roots.

Supplementary problems

A. (c) Let

$$L_{ij} = \begin{cases} \sqrt{\frac{i+1}{i}} & j = i \\ -\sqrt{\frac{i-1}{i}} & j = i-1 \\ 0 & j \neq i, i-1 \end{cases} .$$

Then

$$\begin{aligned} \left(LL^{\dagger}\right)_{ik} &= \sum_j L_{ij} L_{kj} = L_{ii} L_{ki} + L_{i,i-1} L_{k,i-1} \\ &= \begin{cases} 0 - \sqrt{\frac{i-1}{i}} \sqrt{\frac{i}{i-1}} & k = i-1 \\ \left(\sqrt{\frac{i+1}{i}}\right)^2 + \left(-\sqrt{\frac{i-1}{i}}\right)^2 & k = i \\ \sqrt{\frac{i+1}{i}} \cdot \left(-\sqrt{\frac{i}{i+1}}\right) + 0 & k = i+1 \\ 0 & |i-k| \geq 2 \end{cases} \\ &= \begin{cases} 2 & k = i \\ -1 & |k-i| = 1 \\ 0 & |k-i| \geq 2 \end{cases} \\ &= (-A)_{ik} . \end{aligned}$$

B. Let $T \in \mathrm{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \left\{ \underline{v} \in V \mid \exists k : p(T)^k \underline{v} = \underline{0} \right\}$.

(a) For a polynomial $q(x)$ we have $xq(x) = q(x)x$. Then $Tq(T) = q(T)T$. In particular, if $\underline{v} \in \mathrm{Ker}\, q(T)$ then $q(T)T\underline{v} = Tq(T)\underline{v} = \underline{0}$ and hence $T\underline{v} \in \mathrm{Ker}\, q(T)$ as well. We assume that V is finite-dimensional, so each V_p is. In particular let $\{\underline{v}_i\}_{i=1}^n \subset V_p$ be a basis, and let k be large enough such that $p(T)^k \underline{v}_i = \underline{0}$ for each i. Then $\mathrm{Span}_F \{\underline{v}_i\}_{i=1}^n \subset \mathrm{Ker}\, p(T)^k$. But $\mathrm{Ker}\, p(T)^k \subset V_p$ by definition, so $V_p = \mathrm{Ker}\, p(T)^k$. It follows that $m_{T \restriction V_p} | p^k$. Since p is irreducible, each divisor of p^k has the form $p^{k'}$ for some $k' \leq k$. If $V_p = \{\underline{0}\}$ then $m_{T \restriction V_p} = 1$. Otherwise, Id_{V_p} is not the zero map so $m_{T \restriction V_p} \neq 1$ and $k' \geq 1$. In any case, $m_T \left(T \restriction V_p\right) = m_T(T) \restriction V_p = 0$ shows that $p^{k'} = m_{T \restriction V_p} | m_T$.

(b) Let $p, q \in F[x]$ be relatively prime (for example p irreducible and not dividing q). We will show that $q(T)$ is invertible on V_p. We have $V_p = \bigcup_{k=0}^{\infty} \mathrm{Ker}(p^k(T))$, so it is enough to show that $q(T)$ is invertible on each $\mathrm{Ker}\left(p^k(T)\right)$. Since q is prime to p, it is prime to p^k for each k. Since $F[x]$ is a PID, there are $\alpha(x), \beta(x) \in F[x]$ such that $\alpha q + \beta p^k = 1$. Then on $U = \mathrm{Ker}(p^k(T))$ we have $1 = \alpha(T \restriction U) q(T \restriction U) + \beta(T \restriction U) p^k(T \restriction U) = \alpha(T \restriction U) q(T \restriction U)$, so $q(T)$ is indeed invertible on $\mathrm{Ker}(p^k(T))$.

Now, let $B \subset F[x]$ be a set of monic irreducibles, and let $W = \sum_{p \in B} V_p$. We need to show the sum is direct. For this, let $\sum_{i=1}^{m} \underline{v}_i = \underline{0}$ be a minimal dependence where $\underline{v}_i \in V_{p_i}$ for some distinct $p_i \in B$. Let k_m be such that $p_m^{k_m}(T)\underline{v}_m = \underline{0}$. We then have

$$\sum_{i=1}^{m-1} p_m^{k_m}(T)\underline{v}_i = \underline{0}.$$

Since $p_m^{k_m}$ is prime to p_i for $i < m$, $p_m^{k_m}(T)$ is invertible on V_{p_i} so $p_m^{k_m}(T)\underline{v}_i \neq \underline{0}$. This contradicts the minimality of the original combination.

(c) Let $W = \bigoplus_{i=1}^{r} V_{p_i}$ and suppose that $Z = V/W$ is non-zero. Since W is T-invariant we have a quotient map $\bar{T}$ on Z. Since V/W is non-zero, we have $1 \neq m_{\bar{T}} | m_T$. In particular, $m_{\bar{T}}$ has some irreducible factor, without loss of generality p_1. Thus let $\underline{v} \in V$ have non-zero image in Z_{p_1}. Then $\prod_{i=2}^{r} p_i^{k_i}(T)$ is invertible in Z_{p_1} so $\prod_{i=2}^{r} p_i^{k_i}(T)\underline{v}$ has non-zero image there. It follows that $\underline{u} = \prod_{i=2}^{r} p_i^{k_i}(T)\underline{v} \notin W$. But $p_1^{k_1}(T)\underline{u} = m_T(T)\underline{u} = \underline{0}$ shows that $\underline{u} \in V_{p_1} \subset W$, a contradiction.

(d) Since $m_{T \restriction V_{p_i}} | m_T$ and has p_i as its unique irreducible divisor, we have $m_{T \restriction V_{p_i}} | p_i^{k_i}$. This $p_i^{k_i}(T \restriction V_{p_i}) = 0$ and $V_{p_i} \subset \mathrm{Ker}\, p_i^{k_i}(T)$. The reverse containment holds by definition. We remark that k_i is the minimal value for which this is true: if $p_i^{k_i - 1}(T)$ vanished in V_{p_i} then $p_i^{k_i - 1}(T)\prod_{j \neq i} p_j^{k_j}(T)$ would vanish in $\bigoplus_{j=1}^{r} V_{p_j} = V$, contradicting the minimality of m_T.

Lior Silberman's Math 412: Problem set 7 (due 2/11/2017)

Practice

P1. Find the characteristic and minimal polynomial of each matrix:

$$\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}5&0&0&0&0&0\\0&4&0&0&0&0\\0&0&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix},\ \begin{pmatrix}5&0&0&0&0&0\\0&2&1&0&0&0\\0&0&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix}.$$

P2. Show that $\begin{pmatrix}0&1&\alpha\\0&0&1\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ are similar. Generalize to higher dimensions.

The Jordan Canonical Form

1. For each of the following matrices, (i) find the characteristic polynomial and eigenvalues (over the complex numbers), (ii) find the eigenspaces and generalized eigenspaces, (iii) find a Jordan basis and the Jordan form.

$$A=\begin{pmatrix}1&2&1&0\\-2&1&0&1\\0&0&1&2\\0&0&-2&1\end{pmatrix},\ B=\begin{pmatrix}0&1&-1&0\\0&0&0&-1\\1&0&0&1\\0&1&0&0\end{pmatrix},\ C=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&-1&1\\-1&-1&1&1&-1&1\\0&0&0&1&0&0\\0&1&0&0&2&0\\0&0&0&0&0&1\end{pmatrix}.$$

RMK I suggest computing by hand first even if you later check your answers with a CAS.

2. Suppose the characteristic polynomial of T is $x(x-1)^3(x-3)^4$.
   (a) What are the possible minimal polynomials?
   (b) What are the possible Jordan forms?

3. Let $T,S\in\mathrm{End}_F(V)$.
   (a) Suppose that T,S are similar. Show that $m_T(x)=m_S(x)$.
   (b) Prove or disprove: if $m_T(x)=m_S(x)$ and $p_T(x)=p_S(x)$ then T,S are similar.

4. Let F be algebraically closed of characteristic zero. Show that every $g\in\mathrm{GL}_n(F)$ has a square root, in that $g=h^2$ for some $h\in\mathrm{GL}_n(F)$.

5. Let V be finite-dimensional, and let $\mathcal{A}\subset\mathrm{End}_F(V)$ be an F-subalgebra, that is a subspace containing the identity map and closed under multiplication (composition). Suppose that $T\in\mathcal{A}$ is invertible in $\mathrm{End}_F(V)$. Show that $T^{-1}\in\mathcal{A}$.

(extra credit problem on reverse)

Extra credit

6. (The additive Jordan decomposition) Let V be a finite-dimensional vector space, and let $T \in \mathrm{End}_F(V)$.

   DEF An *additive Jordan decomposition* of T is an expression $T = S + N$ where $S \in \mathrm{End}_F(V)$ is diagonable, $N \in \mathrm{End}_F(V)$ is nilpotent, and S, N commute.

   (a) Suppose that F is algebraically closed. Separating the Jordan form into its diagonal and off-diagonal parts, show that T has an additive Jordan decomposition.

   (b) Let $S, S' \in \mathrm{End}_F(V)$ be diagonable and suppose that S, S' commute. Show that $S + S'$ is diagonable.

   (c) Show that a nilpotent diagonable linear transformation vanishes.

   (d) Suppose that T has two additive Jordan decompositions $T = S + N = S' + N'$. Show that $S = S'$ and $N = N'$.

Supplementary problems: ℓ^p spaces

A. For $\underline{v} \in \mathbb{C}^n$ and $1 \le p \le \infty$ let $\|\underline{v}\|_p$ be as defined in class.

   (a) For $1 < p < \infty$ define $1 < q < \infty$ by $\frac{1}{p} + \frac{1}{q} = 1$ (also if $p = 1$ set $q = \infty$ and if $p = \infty$ set $q = 1$). Given $x \in \mathbb{C}$ let $y(x) = \frac{\bar{x}}{|x|}|x|^{p/q}$ (set $y = 0$ if $x = 0$), and given a vector $\underline{x} \in \mathbb{C}^n$ define a vector $\underline{y}$analogously.

      (i) Show that $\left\|\underline{y}\right\|_q = \|\underline{x}\|_p^{p/q}$.

      (ii) Show that for this particular choice of vy, $|\sum_{i=1}^n x_i y_i| = \|\underline{x}\|_p \left\|\underline{y}\right\|_q$

   (b) Now let $\underline{u}, \underline{v} \in \mathbb{C}^n$ and let $1 \le p \le \infty$. Show that $|\sum_{i=1}^n u_i v_i| \le \|\underline{u}\|_p \|\underline{v}\|_q$ (this is called *Hölder's inequality*).

   (c) Conlude that $\|\underline{u}\|_p = \max\left\{|\sum_{i=1}^n u_i v_i| \mid \|\underline{v}\|_q = 1\right\}$.

   (d) Show that $\|\underline{u}\|_p$ is a seminorm (hint: A(c)) and then that it is a norm.

   (e) Show that $\lim_{p\to\infty} \|\underline{v}\|_p = \|\underline{v}\|_\infty$ (this is why the supremum norm is usually called the L^∞ norm).

B. Let X be a set. For $1 \le p < \infty$ set $\ell^p(X) = \{f : X \to \mathbb{C} \mid \sum_{x\in X} |f(x)|^p < \infty\}$, and also set $\ell^\infty(X) = \{f : X \to \mathbb{C} \mid f \text{ bounded}\}$.

   (a) Show that for $f \in \ell^p(X)$ and $g \in \ell^q(X)$ (q as in A(a)) we have $fg \in \ell^1(X)$ and $|\sum_{x\in X} f(x)g(x)| \le \|f\|_p \|g\|_q$.

   (b) Show that $\ell^p(X)$ are subspaces of $\mathbb{C}^X$, and that $\|f\|_p = \left(\sum_{x\in X} |f(x)|^p\right)^{1/p}$ is a norm on $\ell^p(X)$

   (c) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that for each $x \in X$, $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence.

   (d) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence and let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f \in \ell^p(X)$.

   (e) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that it is convergent in $\ell^p(X)$.

Hint for B(d): Suppose that $\|f\|_p = \infty$. Then there is a finite set $S \subset X$ with $\left(\sum_{x\in S} |f(x)|^p\right)^{1/p} \ge \lim_{n\to\infty} \|f_n\| + 1$.

Lior Silberman's Math 412: Solutions to Problem set 7

Practice

P1. For $A=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $A-I=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\frac{1}{2}\exp(it)+\frac{1}{2}\exp(-it)\\0&0&0&0\end{pmatrix}$ so $(A-I)^2=0$ and the minimal polynomial is $(x-1)^2$. The characteristic polynomial must then be $(x-1)^4$.

For $B=\begin{pmatrix}5&0&0&0&0&0\\0&4&0&0&0&0\\0&0&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix}$ we have $V_5=\mathrm{Span}\{\underline{e}_1\}$, $V_4=\mathrm{Span}\{\underline{e}_2\}$, $V_2=\mathrm{Span}\{\underline{e}_3,\ldots,\underline{e}_6\}$ $(B-5,B-4,(B-2)^2$ vanish on the respective spaces, and they sum to F^6). The minimal polynomial is therefore $(x-5)(x-4)(x-2)^2$. The characteristic polynomials on the respective spaces are $(x-5),(x-4),(x-2)^4$ so on their direct sum is $(x-5)(x-4)(x-2)^4$.

For $C=\begin{pmatrix}5&0&0&0&0&0\\0&2&1&0&0&0\\0&0&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix}$ we have $V_5=\mathrm{Span}\{\underline{e}_1\}$, $V_2=\mathrm{Span}\{\underline{e}_2,\ldots,\underline{e}_6\}$ $(C-5,(C-2)^3$ vanish on the respective spaces, and they sum to F^6). The minimal polynomial is then $(x-5)(x-2)^3$ and the characteristic polynomial $(x-5)(x-2)^5$.

P2. Let $N=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$. Then $N\underline{e}_1=\underline{0}$, $N\underline{e}_2=\underline{e}_1$ and $N(\underline{e}_3+\alpha\underline{e}_2)=\underline{e}_2+\alpha\underline{e}_3$. Clearly $\{\underline{e}_1,\underline{e}_2,\underline{e}_3+\alpha\underline{e}_2\}$ is another basis for F^3, so N is similar to $A=\begin{pmatrix}0&1&\alpha\\0&0&1\\0&0&0\end{pmatrix}$. More generally, let A be a strictly upper-triangular matrix with non-zero entries right above the main diagonal. Then A is similar to the Jordan block N of the same size (ones above the main diagonal, zeroes elsewhere). For this let $\underline{v}_n=\underline{e}_n$, and for $0\le k\le n-1$ set $\underline{v}_{n-k}=A^k\underline{v}_n$. We show by induction on k that $\underline{v}_{n-k}\in\mathrm{Span}\{\underline{e}_i\}_{i=1}^{n-k}$ and that the coefficient of $\underline{e}_k$ is the product $\prod_{j=1}^{k}a_{n-j,n-j+1}$. For $k=0$ the claim is evident. Suppose the claim for k. Since A is strictly upper-triangular, we have $A\underline{e}_m\in\mathrm{Span}\{\underline{e}_i\}_{i=1}^{m-1}$. Thus if

$$\underline{v}_{n-k}=\left(\prod_{j=1}^{k}a_{n-j,n-j+1}\right)\underline{e}_{n-k}+\sum_{i=1}^{n-k-1}\alpha_i\underline{e}_i$$

then

$$\begin{aligned}
\underline{v}_{n-k-1} &= \left(\prod_{j=1}^{k} a_{n-j,n-j+1}\right) A\underline{e}_{n-k} + \sum_{i=1}^{n-k-1} \alpha_i A\underline{e}_i \\
&\in \left(\prod_{j=1}^{k} a_{n-j,n-j+1}\right) \sum_{i=1}^{n-k-1} a_{i,n-k}\underline{e}_i + \operatorname{Span}\left\{\underline{e}_j\right\}_{j=1}^{n-k-2} \\
&= \left(\prod_{j=1}^{k} a_{n-j,n-j+1}\right) a_{n-k-1,n-k}\underline{e}_{n-k-1} + \operatorname{Span}\left\{\underline{e}_j\right\}_{j=1}^{n-k-2} \\
&= \left(\prod_{j=1}^{k+1} a_{n-j,n-j+1}\right) \underline{e}_{n-k-1} + \operatorname{Span}\left\{\underline{e}_j\right\}_{j=1}^{n-k-2}
\end{aligned}$$

which is the claim for $k+1$. Since the $a_{i,i+1}$ are non-zero it follows that $\underline{v}_1 = \left(\prod_{i=1}^{n-1} a_{i,i+1}\right)\underline{e}_1$ is non-zero while $A\underline{v}_1 = \underline{0}$, so $\{\underline{v}_i\}_{i=1}^{n}$ is a Jordan block for A and A is similar to N.

1. (a)

$$\begin{aligned}
\det(x\operatorname{Id}-A) &= \det\begin{pmatrix} x-1 & -2 & -1 & 0 \\ 2 & x-1 & 0 & -1 \\ 0 & 0 & x-1 & -2 \\ 0 & 0 & 2 & x-1 \end{pmatrix} = \det\begin{pmatrix} \begin{pmatrix} x-1 & -2 \\ 2 & x-1 \end{pmatrix} & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \\ & \begin{pmatrix} x-1 & -2 \\ 2 & x-1 \end{pmatrix} \end{pmatrix} \\
&= \left(\det\begin{pmatrix} x-1 & -2 \\ 2 & x-1 \end{pmatrix}\right)^2 = \left((x-1)^2+4\right)^2 = \left(x^2-2x+5\right)^2 \\
&= (x-\lambda)^2\left(x-\bar{\lambda}\right)^2
\end{aligned}$$

where $\lambda = 1+2i$. We find some eigenvectors:

$$A - \lambda\operatorname{Id} = \begin{pmatrix} -2i & 2 & 1 & 0 \\ -2 & -2i & 0 & 1 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{pmatrix}$$

so its eigenvectors must take the form (x,y,z,iz) where $-2ix+2y+z=0$, so $(x, ix-z/2, z, iz)$ that is

$$V_\lambda \supset \operatorname{Span}_{\mathbb{C}}\left\{\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ i \end{pmatrix}\right\}.$$

Taking complex conjugates we find

$$V_{\bar{\lambda}} \supset \operatorname{Span}_{\mathbb{C}}\left\{\begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ -i \end{pmatrix}\right\}.$$

Since the whole space is 4-dimensional, we have the eigenbasis $\left\{\begin{pmatrix}1\\i\\0\\0\end{pmatrix},\begin{pmatrix}0\\-1/2\\1\\i\end{pmatrix},\begin{pmatrix}1\\-i\\0\\0\end{pmatrix},\begin{pmatrix}0\\-1/2\\1\\-i\end{pmatrix}\right\}$ so that

$$A = SDS^{-1}$$

where $S=\begin{pmatrix}1&0&1&0\\i&-1/2&-i&-1/2\\0&1&0&1\\0&i&0&-i\end{pmatrix}$ and $D=\operatorname{diag}(1+2i,1+2i,1-2i,1-2i)$.

(b) $\det(\mathrm{Id}-Bx)=\det\begin{pmatrix}x&-1&1&0\\0&x&0&1\\-1&0&x&-1\\0&-1&0&x\end{pmatrix}=x\begin{vmatrix}x&&1\\&x&-1\\-1&&x\end{vmatrix}-\begin{vmatrix}-1&1&1\\x&0&-1\\-1&0&x\end{vmatrix}=x^2\begin{vmatrix}x&-1\\&x\end{vmatrix}+$ $x\begin{vmatrix}&x\\-1&x\end{vmatrix}+\begin{vmatrix}x&1\\-1&x\end{vmatrix}=x^4+x^2+\left(x^2+1\right)=\left(x^2+1\right)^2$. The eigenvalues are therefore $\pm i$. We have

$$B-i\,\mathrm{Id}=\begin{pmatrix}-i&1&-1&0\\0&-i&0&-1\\1&0&-i&1\\0&1&0&-i\end{pmatrix}.$$

Row reduction gives:

$$B-i\,\mathrm{Id}\sim\begin{pmatrix}0&1&0&i\\0&-i&0&-1\\1&0&-i&1\\0&0&0&0\end{pmatrix}\sim\begin{pmatrix}0&1&0&i\\0&0&0&-2\\1&0&-i&1\\0&2&0&0\end{pmatrix}\sim\begin{pmatrix}0&1&0&0\\0&0&0&-2\\1&0&-i&0\\0&0&0&0\end{pmatrix}$$

so $V_i\supset\mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}i\\0\\1\\0\end{pmatrix}\right\}$ and similarly $V_{-i}\supset\mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}-i\\0\\1\\0\end{pmatrix}\right\}$. Now $(B-i)^2=\begin{pmatrix}-2&-2i&2i&-2\\0&-2&0&2i\\-2i&2&-2&-2i\\0&-2i&0&-2\end{pmatrix}$ so $(B-i)^2\begin{pmatrix}0\\i\\0\\1\end{pmatrix}=\underline{0}$ also. Since $(B-i)\begin{pmatrix}0\\i\\0\\1\end{pmatrix}=\begin{pmatrix}i\\0\\1\\0\end{pmatrix}$ so $V_i\supset\mathrm{Span}\left\{\begin{pmatrix}i\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\i\\0\\1\end{pmatrix}\right\}$.

Similarly $V_{-i}\supset\mathrm{Span}\left\{\begin{pmatrix}-i\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\-i\\0\\1\end{pmatrix}\right\}$ and since the whole space is 4-dimensional we conclude that $\mathrm{Span}\left\{\begin{pmatrix}i\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\i\\0\\1\end{pmatrix}\right\}$, $\mathrm{Span}\left\{\begin{pmatrix}-i\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\-i\\0\\1\end{pmatrix}\right\}$ as the two Jordan block.