Lior Silberman's Set Theory: Problem set 4 on cardinality

1. Call a set A *Dedekind–infinite* if it is equinumerous with a proper subset, *Dedekind–finite* otherwise.
   (a) Show that a set is Dedekind–finite iff its satisfies the pigeon–hole principle: every injection $A \to A$ is surjective.
   (b) Show that a subset of a Dedekind–finite set is Dedekind–finite.
   (c) Show that the union of two Dedekind–finite sets is Dedekind–finite.
   (d) Show that the Cartesian product of two Dedekind–finite sets is Dedekind–finite.
   HARD Suppose that there exists an infinite Dedekind–finite set. Show that there exists a Dedekind-finite set A so that every $a \in A$ is Dedekind–finite yet $\bigcup A$ is Dedekind–infinite.

2. Show that every subset of ω is either finite or equinumerous with ω (hint: for a set A define $f(a) = (A \cap a)$).

3. (Countability)
   (a) Show that the set of finite subsets of ω is countable.
   (b) Show that the set of algebraic numbers is countable.

4. Show that for any non-empty set A, there is no set B so that $x \in B$ iff $x \approx A$.

5. Suppose we have a one-to-one function $f\colon A \to B$. Show that there is a surjective function $g\colon B \to A$.