Lior Silberman's Set Theory: Problem Set 6 on ordinals and cardinals

1. Let F be a field, and let V be a vector space over the field F. Let $B, B' \subset V$ be F-bases. Show that $B \approx B'$. (hint: compare B' and the set of finite sequences of elements of B).

DEFINITION. A *Hilbert space* is a real (or complex) vector space X equipped with a (hermitian, in the second case) inner product which is *complete* with respect to the associated Euclidean norm. An *orthonormal system* in X is a subset B so that for any $x, y \in B$ we have $\langle x, y\rangle = \delta_{x,y}$ (Kronecker delta). An *orthonormal basis* is an orthonormal system whose span is dense. The *orthogonal complement* of a subset $C \subset X$ is the set $C^\perp = \{y \in X \mid \forall x \in C : \langle x, y\rangle = 0\}$. Recall the *Cauchy–Schwarz inequality*: for all $x, y \in X$ we have

$$|\langle x, y\rangle| \le \|x\| \, \|y\| = \langle x, x\rangle^{1/2} \langle y, y\rangle^{1/2} .$$

2. Let X be a Hilbert space
   (a) Show that an orthonormal system in X is linearly independent.
   (b) Let $C \subset X$. Show that $C^\perp = (\operatorname{Span} C)^\perp$ is a closed subspace of X.
   (c) Let $B \subset X$. Show that $\operatorname{Span} B \subset X$ is dense iff $B^\perp = \{0\}$.
   (d) (Bessel's inequality) Let $x \in X$ and let $F \subset X$ be a finite orthonormal system. Show that $\sum_{v \in F} |\langle v, x\rangle|^2 \le \|x\|^2$ (hint: let $\alpha_v = \langle v, x\rangle$ and show that $x - \sum_{v \in F} \alpha_v v \in F^\perp$).
   (e) Let $B \subset X$ be an arbitrary orthonormal system and let $x \in X$. Show that $\{v \in B \mid \langle v, x\rangle \ne 0\}$ is countable.
   (f) Let $B \subset X$ be an arbitrary orthonormal system and let $x \in X$. For finite $F \subset B$ let $x_F = \sum_{v \in F} \alpha_v v$. Show that $\{x_F\}_F$ is a Cauchy sequence in the strong sense that, for any $\varepsilon > 0$, there is a finite set F_ε such that if $F, F' \supset F_\varepsilon$ then $\|x_F - x_{F'}\| \le \varepsilon$. Since X is complete it follows that $\sum_{v \in B} \alpha_v v$ converges (note that there are at most countably many non-zero terms in the sum!).
   (g) Let $B \subset X$ be a complete orthonormal system and let $x \in X$. Show that $\sum_{v \in B} \langle v, x\rangle v = x$ (hint: show that $x - \sum_{v \in B} \langle v, x\rangle v \in B^\perp$).
   (h) Let B' be another complete orthonormal system in X. Show that $B \approx B'$ (hint: you need to separate the finite-dimensional and infinite-dimensional cases).

3. Let $(A, <)$ be a linearly ordered set. Call an element $a \in A$ the *successor* of $b \in A$ (and write $a = b^+$) if a is the least element greater than b, that is if $b < a$ and there is no $c \in A$ such that $b < c < a$. Call $a \in A$ a *limit element* if $a = \sup\{b \in A \mid b < a\}$. Show that every element is either a successor or a limit element.

4. Let $(A, <)$ be a linearly ordered set. Show that the following are equivalent:
   (i) A is well-ordered: every non-empty subset of A has a least element.
   (ii) Transfinite induction 1 works on A: suppose $S \subset A$ has the property that if $t \in A$ has $\operatorname{seg} t \subset S$ then $t \in S$. Then $S = A$.
   (iii) Transfinite induction 2 works on A: suppose $S \subset A$ has the propreties (1) if $a \in S$ then $a^+ \in S$ (2) if $a \in A$ is a limit element and $\operatorname{seg} a \subset S$ then $a \in S$. Then $S = A$.