Lior Silberman's Set Theory: Problem Set 7

1. Let $\gamma(x)$ be the formula "x is grounded", that is the formula "there exists α such that $x \subset V_\alpha$". The *relativization* of a formula φ is the formula φ^γ where every quantifier $\forall x$ or $\exists x$ in φ is replaced by a *relative quantifier* $\forall^\gamma x = \forall x : \gamma(x) \wedge$ and $\exists^\gamma x = \exists x : \gamma(x) \wedge$ (in other words, we replace "for all/there exists x" by "for all/there exists *grounded* x".
   (a) Show that the class γ is transitive: $\forall x \forall y \left((x \in y \wedge \gamma(y)) \to \gamma(x) \right)$.
   SCHOLIUM You have shown that "set membership" means the same thing in the "relative world" where we only live in the universe of grounded sets.
   (b) Show that $\mathrm{rank}(\alpha) = \alpha$
   SCHOLIUM You have shown that a "relative ordinal" means the same thing as an an ordinal.
   (c) For each axiom of set theory show that its relativized form holds as well.
   SCHOLIUM You have shown that the class γ is an "*inner model*", having the same ordinals.

2. Suppose we have formulas $\sigma(x), \lambda(x,y)$ so that λ defines a well-ordering on the class σ

$$\begin{aligned} \forall x \forall y \forall z : [& (\sigma(x) \to \neg\lambda(x,x)) \wedge \\ & ((\sigma(x) \wedge \sigma(y) \wedge \sigma(z) \wedge \lambda(x,y) \wedge \lambda(y,z)) \to \lambda(x,z)) \\ & \wedge ((\sigma(x) \wedge \sigma(y)) \to (\lambda(x,y) \vee \lambda(y,x) \vee x = y)] \,, \end{aligned}$$

   and

$$\forall A : ((\exists x \in A \wedge \forall x \in A : \sigma(x)) \to \exists z \in A \forall x \in A : x \neq z \to \gamma(z,x)) \,.$$

   (a) Suppose $S = \{x \mid \sigma(x)\}$ is a set. Show that $R = \left\{ \langle x,y \rangle \in S^2 \mid \gamma(x,y) \right\}$ is a well-ordering of S and the well-ordered set $\langle S,R \rangle$ is order-isomorphic to a unique ordinal.
   (b) Suppose to the contrary that σ defines a proper class. Show that there is a formula $\iota(x,y)$ defining an order isomorphism between the class of all ordinals and (σ, λ).
   (c) Show that any subclass of the ordinals is well-ordered by membership. Conclude that any unbounded subclass of the ordinals is order-isomorphic to the class of all ordinals.
   (d) Apply (c) to the class of all cardinals to obtain an order-preserving formula representing a function $\alpha \mapsto \aleph_\alpha$ enumerating the cardinals in order. Show that this function agrees with the alephs as defined in the textbook.

3. An integer $N \geq 1$ is *completely written in base* $b \geq 2$ when it is written in the form $N = a \cdot b^m + N'$ where (1) m is the largest integer such that $b^m < N$; (2) $1 \leq a < b$ is largest such that $a \cdot b^m < N$; (3) the numbers m, N' are themselves completely written in base b (also declare that 0 has the *empty representation* in base b)
   (a) Show that $m, N' < N$ so that this is a definition by recursion.
   (b) Write 567 completely in base 3 and base 4.
   DEF For a number N and two bases $b' > b$, a *replacement step* consists of writing N completely in base b and then replacing every occurence of b with b'.
   (c) Show that the resulting expression after a replacement step is a number written completely in base b'.
   (**d) Fix an increasing sequences of bases $b_0 < b_1 < b_2 < b_3 < b_4 \ldots$. Given an integer $N \geq 0$ define a sequence of integers as follows: $N_0 = N$, and for $i \geq 1$ given N_i write it completely in base b_i, replace b_i with b_{i+1} to obtain a number N'_{i+1}, and then let $N_{i+1} = N'_{i+1} - 1$. Show that there is a finite j such that $N_i = 0$ for $i \geq j$.