Lior Silberman's Math 223: Problem Set 4 (due 8/2/2021)

Practice problems (recommended, but do not submit)

Section 2.1, Problems 1-3,5,9,10-12,28-29
Section 2.2, Problems 1-3.

Calculations with linear maps

1. Let $T\colon U \to V$ be a linear map, and let $S \subset U$ be a spanning set. Show that $\{T\underline{s} \mid \underline{s} \in S\}$ spans $\operatorname{Im} T$.

RMK This is one starting point for finding a basis for $\operatorname{Im} T$.

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 \end{pmatrix}$.
   (a) Find bases for $\operatorname{Ker} T$, $\operatorname{Im} T$ and check that the dimension formula holds.
   (b) Find the matrix for T with respect to the bases $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ of $\mathbb{R}^2$ and $\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ of $\mathbb{R}^3$.

3. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 - x_2 + x_3 - x_5 \\ -3x_1 - x_3 + x_5 \end{pmatrix}$.
   (a) Find bases for $\operatorname{Ker} T$, $\operatorname{Im} T$ (use problem 1) and check that the dimension formula holds.
   (b) Find the matrix for T with respect to the standard bases of $\mathbb{R}^5$, $\mathbb{R}^3$.
   (c) Find the matrix for T with respect to the standard basis of $\mathbb{R}^5$ and the basis $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}$ of $\mathbb{R}^3$.

4. Let $D\colon \mathbb{R}[x]^{\le n} \to \mathbb{R}[x]^{\le n}$ be the differentiation map.
   (a) Find $\operatorname{Ker} D$ and its dimension.
   (b) Find $\operatorname{Im} D$.
   Fix a number $a \neq 0$ and let $T\colon \mathbb{R}[x]^{\le n} \to \mathbb{R}[x]^{\le n}$ be the map $D + Z_a$ (that is, $Tp = \frac{dp}{dx} + a \cdot p$).
   (c) Show that T maps the basis of monomials to a set of $n+1$ polynomials of distinct degrees.
   (*d) Show that $\operatorname{Im} T = \mathbb{R}[x]^{\le n}$.

5. Write $C^\infty(\mathbb{R})$ for the space of infinitely differentiable functions (i.e. the functions for which derivatives of all orders exist).
   PRAC For a function $a \in C^\infty(\mathbb{R})$ write M_a for the operator of *multiplication by a:* $(M_a f)(x) = a(x)f(x)$. Show that $M_a\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a linear map.
   DEF The *commutator* of two linear maps $A, B\colon V \to V$ is the map $[A,B] = AB - BA$ (in other words $[A,B]\,\underline{v} = A(B(\underline{v})) - B(A(\underline{v}))$).
   (a) Show that $[A,B]$ is a linear map $V \to V$.
   (b) Let $a \in C^\infty(\mathbb{R})$. Find a function b so that $[D, M_a] = M_b$ as linear maps on $C^\infty(\mathbb{R})$.

Linear dependence of functions

6. Let X be a set, and let $\{f_i\}_{i=1}^n \subset \mathbb{R}^X$ be some n functions. Let $\{x_j\}_{j=1}^m \subset X$ be m points of X.
   (a) Define a map $E\colon \mathbb{R}^n \to \mathbb{R}^m$ by setting $(E\underline{a})_j = \sum_{i=1}^n a_i f_i(x_j)$ for $\underline{a} \in \mathbb{R}^n$ and $1 \le j \le m$. Show that E is linear.
   (b) Suppose that $m < n$. Show that $\dim \operatorname{Ker} E > 0$. Conclude that if $m < n$ there exist $\{a_i\}_{i=1}^n$ not all zero such that the function $\sum_{i=1}^n a_i f_i$ vanishes at all the points $\{x_j\}_{j=1}^m$.

Surjective and injective maps; Invertibility

DEFINITION. Let $T\colon U \to V$ be a linear map. We say that T is *injective* (a *monomorphism*) if $T\underline{u} = T\underline{u}'$ implies $\underline{u} = \underline{u}'$ and *surjective* (an *epimorphism)* if $\operatorname{Im} T = V$.

7. Show that T is injective if and only if $\operatorname{Ker} T = \{\underline{0}\}$. (Hint: to compare two vectors consider their difference)

DEFINITION. If a linear map $T\colon U \to V$ is surjective and injective we say it is an *isomorphism* (of vector spaces). We say that U, V are isomorphic if there is an isomorphism between them.

8. Suppose that $T\colon U \to V$ is an isomorphism of vector spaces, and define a function $T^{-1}\colon V \to U$ by $T^{-1}\underline{v}$ being that vector $\underline{u}$ such that $T\underline{u} = \underline{v}$.
   (a) Explain why $\underline{u}$ exists and why it is unique (that is, review the definitions of surjective and injective)
   (*b) Show that T^{-1} is a linear function.