Lior Silberman's Math 223: Problem Set 11 (due 7/4/2021)

Practice problems

Section 6.1: all problems are suitable

A. Write down some matrix $A \in M_4(\mathbb{R})$ such that A has four distinct eigenvalues (your choice) with the correspoding eigenvectors being $\begin{pmatrix}1\\2\\0\\3\end{pmatrix}, \begin{pmatrix}2\\4\\1\\6\end{pmatrix}, \begin{pmatrix}2\\2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}$.

B. Let V be a vector space, $\varphi \in V^*$ a linear functional and $\underline{w} \in V$ a fixed vector. Suppose that $\varphi(\underline{w}) \neq 0$.
  (a) Show directly that $V = \operatorname{Ker}\varphi \oplus \operatorname{Span}(\underline{w})$.
  (b) Show that the map $T: V \to V$ given by $T\underline{v} = \underline{v} - 2\frac{\varphi(\underline{v})}{\varphi(\underline{w})}\underline{w}$ is linear.
  (c) What is T^2?
  (d) Find all the eigenvalues of T (suppose that $\dim V \geq 2$).
  (e) Show that T is diagonable.

More on diagonalization

1. (a) Let V be a real vector space of odd dimension. Prove that every $T \in \operatorname{End}(V)$ has a real eigenvalue.
  (b) Define $T: \mathbb{R}[x]^{\leq 3} \to \mathbb{R}[x]^{\leq 3}$ by $(Tp)(x) = x^3 p(-1/x)$. Prove that T has no real eigenvalues. (Hint: what is T^2?)
  (c) Define $T: \mathbb{C}[x]^{\leq 3} \to \mathbb{C}[x]^{\leq 3}$ by $(Tp)(x) = x^3 p(-1/x)$. Find the spectrum of T and exhibit one eigenvector for each eigenvalue.

2. Let V be a vector space, let $\{\lambda_i\}_{i=1}^r$ be *distinct* numbers, and let $T \in \operatorname{End}(V)$ satisfy $p(T) = 0$ where $p(x) = (x-\lambda_1)\cdots(x-\lambda_r) = \prod_{i=1}^r (x-\lambda_i)$.
  (a) Show that the spectrum of T is contained in $\{\lambda_i\}_{i=1}^n$.
  (b) Fix j and define an auxiliary map $R = R_j$ by $R = \prod_{i\neq j}\left(\frac{T-\lambda_i}{\lambda_j-\lambda_i}\right)$. Show that $T \cdot R = \lambda_j R$.
  (c) Show by induction on k that $T^k R = \lambda_j^k R$ for all $k \geq 0$.
  (d) Show that the maps $q \mapsto q(T)R$, $q \mapsto q(\lambda_i)R$ are linear maps $\mathbb{R}[x] \to \operatorname{End}(V)$. Then show that they are equal.
  (e) Show that R is a projection.
  (f) Show that $\operatorname{Im}(R) = \operatorname{Ker}(T-\lambda_j)$.
  (g) Show that T is diagonable.

3. Fix a vector space V and let $T, S \in \operatorname{End}(V)$ satisfy $TS = ST$.
  (a) Suppose that $T\underline{v} = \lambda\underline{v}$ for some λ and $\underline{v} \in V$. Show that $T(S\underline{v}) = \lambda(S\underline{v})$.
  CONCLUSION Let $V_\lambda = \{\underline{v} \in V \mid T\underline{v} = \lambda\underline{v}\}$. Then $S(V_\lambda) \subset V_\lambda$.
  SUPP Let A, B be invertible linear maps. Show that $AB = BA$ iff $ABA^{-1}B^{-1} = \mathrm{Id}$.
  DEF *An image of the discrete Heisenberg group* is a triple of invertible maps $A, B, Z \in \operatorname{End}(V)$ such that $ABA^{-1}B^{-1} = Z$ and such that $AZA^{-1}Z^{-1} = BZB^{-1}Z^{-1} = \mathrm{Id}$ ("A, B commute with their commutator"). Fix such a triple for the rest of the problem.
  (*b) Let ζ be an eigenvalue of Z, and let λ be an eigenvalue of the map $A\restriction_{V_\zeta}$ we bound in problem (a) (we set $V_\zeta = \operatorname{Ker}(Z-\zeta)$). Show that $\lambda\zeta$ is also an eigenvalue of $A\restriction_{V_\zeta}$ (hint: try doing something to the eigenvector).
  (c) Suppose V is finite-dimensional. Show that we must have $\zeta^k = 1$ for some k.
  (d) Compute $\det(Z\restriction_{V_\zeta})$ in two different ways to show that $\zeta^{\dim V_\zeta} = 1$.

Calculating with inner products

4. Let $S = \left\{ \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i+1 \\ 1-2i \end{pmatrix}, \begin{pmatrix} 0 \\ 5+2i \\ 1+2i \end{pmatrix} \right\} \subset \mathbb{C}^3$.
   (a) Calculate the 9 pairwise inner products of the vectors.
   (b) Calculate the norms of the three vectors (recall that $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$).

5. Let $S = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$.
   (a) Verify that this is an orthonormal basis of $\mathbb{R}^3$.
   (b) Find the coordinates of the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ in this basis.

6. Find an orthonormal basis for the subspace $W^\perp \subset \mathbb{R}^4$ if $W = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

7. Using the standard (L^2) inner product on $C(-1,1)$ apply the Gram–Schmidt procedure to the following independent sequences:
   (a) $\{1, x, x^2\}$ (in that order)
   RMK Applying the Gram–Schmidt procedure to the full sequence $\{x^n\}_{n=0}^{\infty}$ yields the sequence of *Legendre polynomials* $P_n(x)$ (with a non-standard normalization).
   (b) $\{x^2, x, 1\}$ (in that order)

PRAC In each case apply the Gram–Schmidt procedure to the first few members of the sequence $\{1, x, x^2, \cdots\}$ with respect to the given inner product on $\mathbb{R}[x]$.
   (a) (Hermit polynomials) $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) g(x) e^{-x^2} \, \mathrm{d}x$.
   (b) (Laguerre polynomials) $\langle f, g \rangle = \int_0^{\infty} f(x) g(x) e^{-x} \, \mathrm{d}x$.

Supplementary problems: the minimal polynomial

A. (Division with remainder) Let $p, a \in \mathbb{R}[x]$ with a non-zero. Show that there are unique $q, r \in \mathbb{R}[x]$ with $\deg r < \deg a$ such that $p = qa + r$. (*Hint*: let r be an element of minimal degree in the set $\{p - aq \mid q \in \mathbb{R}[x]\}$).

B. Let V be an n-dimensional vector psace and let $T \in \mathrm{End}(V)$.
   (a) Show that there exists a non-zero $p \in \mathbb{R}[x]^{\leq n^2}$ such that $p(T) = 0$.
   (*Hint*: what is $\dim \mathrm{End}(V)$?)
   DEF A polynomial is *monic* if the highest-degree monomial has coefficient 1 ($x^2 + 3$ is monic, $2x^2 + 3$ is not).
   (b) Rescaling the polynomial, show that there exists a monic polynomial p' of the same degree as p such that $p'(T) = 0$.
   (c) Let $m_T \in \mathbb{R}[x]$ be a monic polynomial of least degree such that $m_T(T) = 0$. Show that for any $p \in \mathbb{R}[x]$ we have $p(T) = 0$ iff $m_T \mid p$ in $\mathbb{R}[x]$, that is if there is $q \in \mathbb{R}[x]$ such that $p = m_T q$.
   (d) Let $\tilde{m}_T$ be another monic polynomial of the same degree as m_T such that $\tilde{m}_T(T) = 0$. Show that $\tilde{m}_T = m_T$.
   DEF m_T is called the *minimal polynomial* of T (saying "the" minimal polynomial is justified by part d).
   RMK The Cayley–Hamilton Theorem states that $p_T(T) = 0$ (here p_T is the characteristic polynomial). It follows that $\deg m_T \leq \deg p_T \leq n$ and that $m_A | p_A$.

Supplementary problem: The Rayleigh quotient

C. Given a matrix $A \in M_n(\mathbb{R})$ consider the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(\underline{x}) = \underline{x}^t A \underline{x} = \sum_{i,j=1}^n a_{ij} x_i x_j$. We introduce the notation $\|\underline{x}\|_2^2 = \sum_{i=1}^n x_i^2$.
   (a) Show that $(\nabla f)(\underline{x}) = A\underline{x} + A^t \underline{x}$.
   (b) Let $\underline{v}$ be the point where f attains its maximum on the unit sphere $S^{n-1} = \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\| = 1\}$. Use the method of Largrange multipliers to show that $\underline{v}$ satisfies $A\underline{v} + A^t \underline{v} = \lambda \underline{v}$ for some $\lambda \in \mathbb{R}$.
   (c) A matrix is *symmetric* if $A = A^t$. Show that every symmetric matrix has a real eigenvalue.
   (d) Show that the following two maximization problems are equivalent:

$$\max\{f(\underline{x}) \mid \|\underline{w}\|_2 = 1\} \leftrightarrow \max\left\{\frac{f(\underline{x})}{\|\underline{x}\|_2^2} \mid \underline{x} \neq \underline{0}\right\}.$$