Math 223, Lecture 1

Phenomenon of linearity.

Change of language: Worksheet for Math 223 Lecture 1, 11/1/2021

We study real-valued functions on a fixed interval $[a,b]$.

EXAMPLE. From calculus we know "If f,g are continuous functions and α is a real number then the functions $f+g$ and αf are continuous as well".

Problem: Rewrite this as a statement about the set $C(a,b)$ of continuous functions on the interval.

Solution: The statement literally reads "if $f,g \in C(a,b)$ then $f+g, \alpha f \in C(a,b)$" and it can be rewritten as "The set $C(a,b)$ is closed under addition of functions and under multiplying functions by real numbers".

THEOREM (Calculus). *Let $f,g \in \mathbb{R}^{[a,b]}$ and let $\alpha \in \mathbb{R}$.*

(1) *If f,g are differentiable at x_0 then so are $f+g$ and αf.*
(2) *If f,g are Riemann–integrable on $[a,b]$ then so are $f+g$ and αf.*
(3) *If f,g have finite H^1 Sobolev norm, then so do $f+g$ and αf.*

FACT (Physics). *4. Let ϕ, ψ be possible states of a quantum-mechanical system, and let $a \in \mathbb{C}$. Then $\phi+\psi$ and $a\phi$ are also quantum states.*

PROBLEM. Convert the four claims to statements about the set $D_{x_0}(a,b)$ of functions which are differentiable at x_0, the set $R(a,b)$ of functions which are Riemann-integrable, the Sobolev space $H^1(a,b)$, and the set $\mathcal{H}$ of quantum states. Follow the template above.

(1)

(2) "The set of Riemann-integrable functions is closed under addition & scalar multiplication".

(3)

(4)

Extra credit: Deduce from example 1 and the Theorem a similar statement about the set $C^1(a,b)$ of functions on $[a,b]$ whose derivative is continuous.

EXAMPLE. Let $f,g \in C(a,b)$ be continuous functions and let $\alpha \in \mathbb{R}$. Then at any point $x_0 \in [a,b]$, $(f+g)(x_0) = f(x_0)+g(x_0)$, $(\alpha f)(x_0) = \alpha(f(x_0))$.

Problem: Rewrite this as a statement about the *evaluation map* that takes a function to its values.

Solution: "Evaluation of a sum of function gives the sum of the values; evaluation of a rescaled fuction gives a rescaled value".

THEOREM (Calculus).

(1) If f,g are differentiable at x_0 then
$$(\alpha f+\beta g)'(x_0) = \alpha\left(f'(x_0)\right)+\beta\left(f'(x_0)\right)$$
(2) If f,g are differentiable on $[a,b]$ then
$$(\alpha f+\beta g)' = \alpha f'+\beta g'$$
(3) If f,g are integrable on $[a,b]$ then
$$\int_a^b (\alpha f+\beta g)(x)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x+\beta\int_a^b g(x)\,\mathrm{d}x$$

PROBLEM. In each case identify (a) the space of functions on which the operation is defined; (b) the space of *results* for the operation.

(1)

(2)

(3)

REMARK. A *quantum observable* is a map $\hat{A}$ from the space $\mathcal{H}$ of quantum states to itself such that $\hat{A}(a\phi+b\psi) = a\hat{A}(\phi)+b\hat{A}(\psi)$ for all states ϕ,ψ and all complex numbers a,b.