Math 223, Lecture 19

<u>Last time:</u> $\dim V = n$, $\{\text{volume forms}\} =$

① $= \{ f : V^n \to \mathbb{R} \mid \begin{matrix} f \text{ is } n\text{-linear} \\ + \text{alternating} \end{matrix} \}$

② if f is a volume form, $\underline{u}_i = \sum_j a_{ij} \underline{v}_j$

~~for~~ Then

$$f(\underline{u}_1, \dots, \underline{u}_n) = \sum_{\sigma : \{1,\dots,n\} \mathfrak{S}} (-1)^\sigma \prod_{i=1}^n a_{i\sigma(i)} f(\underline{v}_1, \dots, \underline{v}_n)$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n a_{i\sigma(i)} \; f(\underline{v}_1, \dots, \underline{v}_n)$$

$S_n = \{ \sigma : \{1,\dots,n\} \to \{1,\dots,n\} \mid \begin{matrix} \sigma \text{ is a} \\ \text{bijection} \end{matrix} \}$

$(-1)^\sigma \in \{1, -1\}$ if $\sigma \in S_n$ $\quad ((-1)^\sigma = 0$ otherwise$)$

③ <u>Before last time</u>: for $A \in M_n(\mathbb{R})$, set

$$\det A = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(A^{1j})$$

Cor of 2: $\dim \{ \begin{matrix} \text{space of volume} \\ \text{forms} \end{matrix} \} \le 1$

if f, g volume form, $\{v_j\}_{j=1}^n \subset V$ is a basis, then

$$f(u_1, , u_n) = \left(\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n a_{i\sigma(i)}\right) \cdot f(v_1, , v_n)$$

$$g(u_1, , u_n) = \left(\text{— '' —} \right) \cdot g(v_1, , v_n)$$

(any two non-zero volume forms are proportional).

<u>Theorem</u>: This space <u>is</u> 1d., i.e. the map

$$f \mapsto f(v_1, , v_n) \quad \text{is an isom}$$

$$\{\text{volume forms}\} \to \mathbb{R}$$

<u>Pf</u>: We know the map is injective, just need a nonzero volume form.

We have one: the ad-hoc determinant from before. We show that $A \mapsto \det A$ is a volume form on $\mathbb{R}^n$ (use the n vectors as colums of A)

Need to show: $A \to \det A$ is linear in each column, alternating (two cols equal $\Rightarrow$ det $=0$), non-zero.

Pf: by induction on n. For $n=1$, $\det(a_{11}) = a_{11}$ nothing to prove, ~~sh~~ Jm, $n+1$

Let $A \in M_{n+1}(\mathbb{R})$ $A = (\underline{v}_1, \ldots, \underline{v}_{n+1})$, $v_j \in \mathbb{R}^{n+1}$

suppose kth column has $\underline{v}_k =$ linear combo

$$A = (a_{ij}) \qquad a_{ik} = \beta b_i + \gamma c_i$$

(if $j \neq k$ a_{ij} just a_{ij}, but kth col is a combo)

$$\begin{pmatrix} a_{11} & b_1 + c_1 & a_{13} \\ a_{21} & b_2 + c_2 & a_{23} \\ a_{31} & b_3 + c_3 & a_{33} \end{pmatrix}$$

$$\det A = \sum_{j=1}^{n+1} (-1)^{1+j} a_{1j} \det(A^{1j})$$

Let B = matrix with b_i, C = matrix with c_i

$$B = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \quad C = \begin{pmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{pmatrix}$$

$a_{31} \quad b_3 \quad a_{33} \quad \quad a_{31} \quad c_3 \quad a_{33}$

Warning: not true that $A = \beta B + \gamma C$

Now $\det(A) = \sum_{j=1}^{n+1} (-1)^{1+j} a_{1j} \det(A^{1j})$

$$= \sum_{j \neq k} (-1)^{1+j} a_{1j} \det(A^{1j}) + (-1)^{1+k} a_{1k} \det(A^{1k})$$

if $j \neq k$, kth col still in A^{1j}, still a combo so by induction,

$$\det(A^{1j}) = \beta \det(B^{1j}) + \gamma \det(C^{1j})$$

Also, $A^{1k} = B^{1k} = C^{1k}$, $a_{1k} = \beta b_1 + \gamma c_1$

$$= \sum_{j \neq k} (-1)^{1+j} a_{1j} (\beta \det B^{1j} + \gamma \det C^{1j}) + (-1)^{1+j} (\beta b_1 + \gamma c_1) \cdot \det A^{1k}$$

$$= \beta \left(\sum_{j \neq k} (-1)^{1+j} b_{1j} \det B^{1j} + (-1)^{1+k} b_1 \det B^{1k} \right) + \gamma \left(\sum_{j \neq k} (-1)^{1+j} c_{1j} \det C^{1j} + (-1)^{1+k} c_1 \det C^{1k} \right)$$

$= \beta \det B + \gamma \det C$ as claimed def

Next, suppose kth and lth cols of A are equal. Then same holds for A^{1j} if $j \neq k, l$

so $\det(A) = \sum_{j=1}^{n+1} (-1)^{1+j} a_{1j} \det(A^{1j}) =$

$= (-1)^{1+k} a_{1k} \det(A^{1k})$

$+ (-1)^{1+l} a_{1l} \det(A^{1l})$

$n \times n$ matrices A^{1k}, A^{1l} identical, except in order of cols

$$A = (\ldots, \underline{u}, \ldots, \underline{u}, \ldots)$$

kth lth cols

A^{1k}, A^{1l} = delete one of two $\underline{u}$'s (& first row)

But det on $n \times n$ matrices is alternating (by induction), so anti-symmetric,

say wlog $k < l$

A^{1l} has u in kth column

A^{1k} has $\underline{u}$ in $(l-1)$st column

so to move $\underline{u}$ in A^{1k} to kth col need

$l-1-k$ swaps

original $\left(\underline{u}_1, \underline{u}_2, \cdots\right)$ $\quad \left(\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}\right)$

$$|\underline{u}_2', \underline{u}_3', \underline{u}'| \qquad |\underline{u}_1', \underline{u}_2', \underline{u}_3'|$$

swap first & last. $\longrightarrow$ $\Vert$ $\qquad$ not same

$$-|\underline{u}', \underline{u}_3', \underline{u}_2'|$$

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so $\det(A^{1l}) = (-1)^{l-1+k} \det(A^{1k})$

so $\det A = a_{1k}\left((-1)^{1+k} + (-1)^{l+1} \cdot (-1)^{l-1+k}\right) \det(A^{1k})$

$= a_{1k}\left((-1)^{1+k} + (-1)^{2l+k}\right)$

$= (-1)^k a_{1k}(-1+1) = 0.$ ✓

Last: know ad-hoc det has

$$\det(\mathrm{diag}(a_1, \ldots, a_n)) = a_1 a_2 \cdots a_n$$

this is non-zero $\checkmark$.

Ex: similar argument shows ad-hoc det is a vol form as a function of the rows (see textbook)

Big payoff:

let $T \in \mathrm{End}(V)$. Then the map on space of volume forms

$$f \mapsto Tf \overset{\text{def}}{=} f(T\cdot, T\cdot, \ldots, T\cdot)$$

is linear, so is of the form $Tf = c \cdot f$

Def: Call constant c the determinant of T.

Prop: let A be the matrix of T wrt basis $\{v_j\}_{j=1}^n \subset V$. Then

$$\det T = \det A$$

new def un-noc arg.

Pf: Let f be a non-zero volume form

Then $(Tf)(\underline{v}_1, .., \underline{v}_n) = f(T\underline{v}_1, .., T\underline{v}_n)$

$$= f\left(\left(\sum_{i=1}^{n} a_{ij} \underline{v}_j\right)\right) =$$

$$= (\det A) \cdot f(\underline{v}_1, .., \underline{v}_n)$$

calc from before.

but $(Tf)(\underline{v}_1, .., \underline{v}_n) = (\det T) \cdot f(\underline{v}_1, .., \underline{v}_n)$ ∎

Prop: Let $T, S \in End(V)$

(1) $\det Id_V = 1$

(2) $\det TS = (\det T)(\det S)$

(3) $\det T \neq 0$ iff T is invertible

Pf: Let f be a non-zero volume form

(1) $f(Id_V \cdot \underline{u}_1, .., Id_V \underline{u}_n) \overset{1\cdot}{=} f(\underline{u}_1, .., \underline{u}_n)$

(2) $\det(TS) \cdot f(\underline{u}_1, .., \underline{u}_n) = f(TS\underline{u}_1, .., TS\underline{u}_n)$

$$= f(T(S\underline{u}_1), .., T(S\underline{u}_n))$$

$= (\det T) \cdot f(Su_1, , Su_n) = (\det T)(\det S) \cdot f(u_1, , u_n)$

(3) If T is invertible, then

$$1 = \det(\mathrm{Id}_V) = \det(TT^{-1}) = (\det T)(\det T^{-1})$$

so $\det T \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$

If T not invertible, let $\underline{v}_1 \in \ker T$ be non-zero, complete v_1 to a basis of V.

Then $(\det T) \cdot f(v_1, , v_n) = f(Tv_1, , Tv_n)$

$$= f(\underline{0}, Tv_2, , Tv_n) = 0$$

& $f(v_1, , v_n) \neq 0$ (if $f = 0$ on basis, $f = 0$ everywhere)

so $\det T = 0$

<u>Theorem</u>: let A, B be square matrices related by a sequence of row <u>or</u> column operations.

Then $\det(A) = c \cdot \det(B)$
where c comes from rescale

$\Rightarrow$ use Gauss to compute dets

$$\begin{vmatrix} 8 & 3 & 11 \\ 9 & 1 & 5 \\ 3 & 3 & 3 \end{vmatrix} = \begin{vmatrix} -19 & 0 & -4 \\ 9 & 1 & 5 \\ -26 & 0 & -12 \end{vmatrix} - 3 \times \text{top row}$$

subtract 3 × middle row from top, bottom

$$= \begin{vmatrix} -19 & 0 & -4 \\ 9 & 1 & 5 \\ 31 & 0 & 0 \end{vmatrix} = 31 \begin{vmatrix} -19 & 0 & -4 \\ 9 & 1 & 5 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -31 \begin{vmatrix} 1 & 0 & 0 \\ 9 & 1 & 5 \\ -19 & 0 & -4 \end{vmatrix} = -31 \cdot 1 \cdot \begin{vmatrix} 1 & 5 \\ 0 & -4 \end{vmatrix}$$

minor expansion by first row

$$= -31 \cdot 1 \cdot 1(-4) = 124$$

$$-31 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{vmatrix} = -31 \cdot 1 \cdot 1(-4) = 124$$

upper-triangular