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Math 223, Lecture 21

$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $\qquad$ $(1\,3\,2\,4)$

$$\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} \to \begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 6\\ 3 & 8 & 9\end{pmatrix}$$

$$\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix} = 1\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix} - 2\begin{vmatrix}4 & 6\\ 7 & 8\end{vmatrix} + 3\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}$$

$$\begin{vmatrix}1 & 4 & 7\\ 2 & 5 & 6\\ 3 & 8 & 9\end{vmatrix} = 1\begin{vmatrix}5 & 6\\ 8 & 9\end{vmatrix} - 2\begin{vmatrix}4 & 7\\ 8 & 9\end{vmatrix} + 3\begin{vmatrix}4 & 7\\ 5 & 6\end{vmatrix}$$

<u>Next section</u>: <u>Eigenvalues & Eigenvectors</u>

<u>Preliminary point</u>: change of basis and similarity.

Let T be a linear map $T: V \to V$.

Fix a basis $\{v_i\}_{i=1}^{n} \subset V$. Then the matrix of T in basis v_i is determined by

$$T v_j = \sum_{i=1}^{n} a_{ij} v_i.$$

Let $\{u_k\}_{k=1}^{n} \subset V$ be another basis.

Then T has a matrix wrt this basis too.

How are they related?

say the matrices are

$$T v_j = \sum_i a_{ij} v_i$$

$$T u_\ell = \sum_k b_{k\ell} v_k.$$

expand $v_i = \sum s_{ki} u_k$ (matrix of map $S u_i = v_i$ in $\{u\}$ basis)

Then $\sum_\ell s_{\ell j} T u_\ell = \sum_k \left(\sum_i s_{ki} a_{ij} \right) v_k$

multiply by $(S^{-1})_{jm}$ and sum

$\sum (\sum s_{\ell j} s^{-1}_{jm}) T u_\ell$

$$= \sum_k \left(\sum_i \sum_j s_{ki}\, a_{ij}\, s^{-1}_{jm} \right) \underline{u}_k$$

$$= \sum_\ell \delta_{\ell m}\, T\underline{u}_\ell = T\underline{u}_m$$

$$\Rightarrow T\underline{u}_m = \sum_k \left(\sum_{i,j} s_{ki}\, a_{ij}\, s^{-1}_{jm} \right) \underline{u}_k$$

$$\Rightarrow \quad T\underline{u}_n = \sum_k b_{kn}\, \underline{u}_k$$

$$\Rightarrow B = SAS^{-1}!$$

This is called "change of basis.

If A = matrix of T in basis $\{\underline{v}_i\}$

$\{\underline{u}_k\}$ new basis,

Matrix of T in basis $\{\underline{u}_k\}$ = SAS^{-1}

or $(S^{-1})^{-1} A S^{-1}$, where

$\boxed{\text{Columns of } S = \text{coeffs of } \underline{v}_j \text{ in basis } \{\underline{u}_k\}}$

$\boxed{\text{columns of } S^{-1} = \text{coeff of } \underline{u}_i \text{ in basis } \{\underline{v}_i\}}$

<u>E.g.:</u> start with basis $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

linear map with matrix $A = \begin{pmatrix}26 & -18\\36 & -25\end{pmatrix}$

try basis $\left\{\begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right\}$ instead

$$\begin{pmatrix}3\\4\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}$$

What is the matrix of the linear map in new basis? let $S = \begin{pmatrix}3 & 2\\4 & 3\end{pmatrix}$

① Using formula above, it is

$$\underbrace{\begin{pmatrix}3 & -2\\-4 & 3\end{pmatrix}}_{S^{-1}}\begin{pmatrix}26 & -18\\36 & -25\end{pmatrix}\begin{pmatrix}3 & 2\\4 & 3\end{pmatrix} \leftarrow S$$

cols = new basis in terms of old

$$3\begin{pmatrix}3\\4\end{pmatrix} - 4\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$-2\begin{pmatrix}3\\4\end{pmatrix} + 3\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}.$$

② Directly: $\begin{pmatrix} 26 & -18 \\ 36 & -25 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} 26 & -18 \\ 36 & -25 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

so matrix is $\begin{pmatrix} 2 & \\ & -1 \end{pmatrix}$.

What is the matrix of $(L_A)^{1000}$? it is A^{1000}.

In basis $\left\{\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$, the matrix is $\begin{pmatrix} 2^{1000} & \\ & (-1)^{1000} \end{pmatrix}$

$$= \begin{pmatrix} 2^{1000} & \\ & 1 \end{pmatrix}.$$

so in the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ it is

$S\begin{pmatrix} 2^{1000} & \\ & 1 \end{pmatrix}S^{-1}$ (because we change basis the other way)

$$= \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2^{1000} & \\ & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

<u>Different pov:</u> $A = S^{-1}\underbrace{\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}}_{D}S$

$$A^n = (S^{-1}D\cancel{S})(\cancel{S^{-1}}D\cancel{S})(\cancel{S^{-1}}D\cancel{S})(\cancel{S^{-1}}DS)\cdots$$

$SS^{-1} = Id$

$= S^{-1} D^n S$

Def: Two square matrices / linear maps $A, B : V \to V$ are similar if there is an invertible map $S : V \to V$ s.t.

$$B = S^{-1} A S.$$

just saw similar matrices represent same linear map in different basis

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Eigenvalues:

Diagonal matrices are "easy".

Call a linear map "diagonable" if it has a basis in which the matrix is diagonal

(a matrix is diagonable if it's similar to a diagonal matrix)

Important tool in linear algebra:

"diagonalization"

= finding a basis (if it exists) where

a linear 'map/matrix is diagonal'

<u>Q</u>: What can the matrix of $T: V \to V$ be?

choose a basis $\{\underline{v}_i\} \subset V$ "source",

use $\{T\underline{v}_i\} \subset V$ as basis of "image".

A = matrix of T wrt $\{\underline{v}_i\}, \{\underline{v}_i\}$

AS = " " " " $\{\underline{v}_i\}, \{\underline{u}_j\}$

$S^{-1}AS$ = " " " " $\{\underline{v}_i\}, \{\underline{u}_j\}$

(for AB to be matrix of XY

need same basis ~~used~~ for target of Y, source of X)

$Y: U \to V, \quad X: V \to W.$

<u>later</u>: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ not similar to a diagonal matrix

$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ & \end{pmatrix}$ so $\begin{pmatrix} 1 & x \\ & \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ & \end{pmatrix}.$