Math 223, Lecture 22

Last time: square matrices $A, B \in M_n(\mathbb{R})$ are similar if there is $S \in M_n(\mathbb{R})$, invertible, s.t.

$$B = S^{-1} A S .$$

more generally,

Linear maps $\Phi, \Psi \in End(V)$ are similar if there is $S \in End(V)$ s.t.

$$\Psi = S^{-1} \Phi S$$

Similar matrices represent same linear map in different bases. (S = "change of basis" matrix,

cols of S = coeffs of new basis wrt old basis

Ex: This is an equivalence relation:

(1) A is similar to A

(2) if A is similar to B, B is similar to A,

(3) if A is similar to B and B " " " C, then A is " " C.

A particularity good basis to change to is one where the matrix is diagonal

<u>Key def</u>: Let $T \in End(V)$. Call a number λ an <u>eigenvalue</u> of T if there is a non-zero vector $\underline{v}$ ("eigenvector") s.t.

$$T\underline{v} = \lambda \underline{v}.$$

<u>warning</u>: this is a nonlinear equation!

(but, it is linear in $\underline{v}$ for fixed λ)

<u>Observe</u>: λ is an eigenvalue iff the linear equation $T\underline{v} = \lambda\underline{v}$ has a non-zero solution, i.e iff $Ker(T-\lambda Id) \neq \{\underline{0}\}$.

<u>Why care</u>: Much easier to calculate in a basis of eigenvectors

<u>Eg</u>: Easy to compute T^2, T^{1000} in that basis:

(Ex) if $T\underline{v} = \lambda\underline{v}$ then $T^{1000}\underline{v} = \lambda^{1000}\underline{v}$

so if $T\underline{v}_i = \lambda_i \underline{v}_i$, then

$$T^{1000}\Big(\sum_{i=1}^{r} a_i \underline{v}_i\Big) = \sum_{i=1}^{r} \big(a_i \cdot \lambda_i^{1000}\big)\, \underline{v}_i .$$

Examples: (1) $\{\cos(2\pi kx)\}_{k=0}^{\infty} \cup \{\sin(2\pi kx)\}_{k=1}^{\infty}$

are a "basis" of the space of periodic functions $\{f \mid f(x+1) = f(x)\}$

(checked that they are indep)

so can solve differential equations

by expanding in this basis

(~~Fourier~~ ("Fourier analysis")

$$\frac{d^2}{dx^2}\big(\underbrace{\cos(2\pi kx)}_{\text{"vector"}}\big) = \underbrace{-4\pi^2 k^2}_{\text{number}} \underbrace{\cos(2\pi kx)}_{\text{same vector}}$$

(2) $\frac{d}{dx}(e^{sx}) = s e^{x}$ ("Laplace transform")

can also represent functions as (integrals) linear combos of exponentials.

$$\frac{d^2}{dx^2} f + 7\frac{d}{dx} f + 8f = x^2 \log x$$

write $x^2 \log x = \int_{\gamma} \tilde{h}(s)\, e^{sx}\, ds$

~~then~~ write $f = \int_{\gamma} \tilde{f}(s)\, e^{sx}\, ds$

get $\int_{\gamma} \tilde{f}(s)\,(s^2 + 7s + 8)\, e^{sx}\, ds$

$= \int \tilde{h}(s) \cdot e^{sx}\, ds$

$\Rightarrow \tilde{f}(s) = \frac{1}{s^2 + 7s + 8} \tilde{h}(s)$

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(3) Difference equations : $L : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is the left shift.

F_n = Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$

$$F_{n+2} = F_{n+1} + F_n$$

$\Leftrightarrow (L^2 - L - 1)\, \underline{F} = \underline{0}$.

$\dim \operatorname{Ker}(L^2 - L - 1) = 2$. (If we know F_0, F_1

know Σ)

Let r_1, r_2 be roots of $\lambda^2 - \lambda - 1 = 0$
("characteristic equation")

$$r_1, r_2 = \frac{1 \pm \sqrt{5}}{2}.$$

Then $L((r_i^n)_{n\geq 0}) = (r_1, r_2^2, r_1^3, \dots)$

$$= r_i(1, r_2, r_2^2, \dots) = r_i (r_i^n)_{n\geq 0}$$

so $(L^2 - L - 1)((r_i)^n_{n\geq 0}) = \underbrace{(r_i^2 - r_i - 1)}_{=0} \cdot (r_i^n)_{n\geq 0}$

so have two elements in Ker $\Rightarrow$ basis.

$\Rightarrow$

<u>Thm</u>: If $F_0 = 0$, $F_1 = 1$, then

$$F_n = \frac{1}{\sqrt{5}}\left((r_1)^n_{\geq 0} - (r_2^n)_{n\geq 0}\right)$$

i.e. $$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

<u>Cor</u>: $-1 < \frac{1-\sqrt{5}}{2} < 0$ so F_n is exponentially close to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

so $\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2}$, $\sqrt[n]{F_n} \to \frac{1+\sqrt{5}}{2}$.

Ex: Have population, measure characteristics $t_1, \dots, t_r$

$\Rightarrow$ correlation matrix $C_{ij} = \mathbb{E}(t_i t_j)$

(really should use $\mathbb{E}(t_i - \mu_i)(t_j - \mu_j)$

μ_i = mean of t_i = $\mathbb{E}\, t_i$.

clearly $C_{ij} = C_{ji}$. Later $\Rightarrow$ basis of eigenvectors

say $C \cdot (\underline{v}) = \lambda \underline{v}$ $\underline{v}$ corresponds to a "compound characteristic":

$$t_{\underline{v}} = \sum_i t_i v_i$$

$\lambda \sim$ effect of $t_{\underline{v}}$ on variance.

Numerical example

let $A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$. Want to solve $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

that is $\begin{cases} 4x + 3y = \lambda x \\ x + 2y = \lambda y \end{cases}$

$\Leftrightarrow \begin{cases} (4-\lambda)x + 3y = 0 \\ x + (2-\lambda) y = 0 \end{cases}$

suppose $(x, y; \lambda)$ is a solution

then $x = (\lambda - 2) y$ so

$(4-\lambda)(\lambda-2)y + 3y = 0$

$\Leftrightarrow$

$(\lambda^2 - 6\lambda + 5) y = 0$

<u>Two possibilities:</u> (1) $y = 0$, $\Rightarrow x = 0$
λ arbitrary

(2) $(\lambda^2 - 6\lambda + 5) = 0$ then y is arbitrary,
$x = (\lambda - 2) y$.

$\Leftrightarrow$ if $\lambda^2 - 6\lambda + 5 = 0$, then $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} \lambda - 2 \\ 1 \end{pmatrix}$

$(\lambda - 1)(\lambda - 5)$

So either $\lambda = 1$, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$
or $\lambda = 5$, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$.

Def: The eigenspace corresponding to the eigenvalue λ is $\ker(A - \lambda)$.

Recap: Found λ as the root of the "characteristic polynomial" of the map [non-linear].

Then for each λ, finding eigenspace is purely linear.

Fix v.s. V, $T \in \mathrm{End}(V)$. Then λ is an eigenvalue iff $\ker(T - \lambda \mathrm{Id}) \neq \{0\} \Leftrightarrow \ker(\lambda \mathrm{Id} - T) \neq \{0\}$

If $\dim V = n < \infty$ then non-zero kernel
$\Leftrightarrow$ not invertible ("singular")
$\Leftrightarrow \det(\lambda - T) = 0$

Def: Let V be f.d. The characteristic polynomial of T is the polynomial

$$p_T(x) = \det(x\,\mathrm{Id}_V - T)$$

E.g. $p_{\begin{pmatrix}4&3\\1&2\end{pmatrix}}(x) = \det\begin{pmatrix} x-4 & -3 \\ -1 & x-2 \end{pmatrix} = (x-4)(x-2)-(-1)(-3)$

$$= x^2-6x+8-3 = x^2-6x+5 = (x-1)(x-5)$$

Then λ is an eigenvalue <u>iff</u> $p_T(\lambda)=0$

iff λ is a root of p_T.

Q: What if p_T has no real roots?

<u>Fact</u>: (Fundamental theorem of algebra)

Every real polynomials splits completely over the <u>complex numbers</u>: $\mathbb{C} = \{a+bi \mid a,b\in\mathbb{R}\}$, $i^2=-1$.

<u>Next time</u>: Another proof of existence of eigenvalues, more on complex eigenvalues, ...

<u>Ex</u>: Matrix $N = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ Has $p_N(x) = x^2$

only eigenvalue is 0

0 is a double root ("alg. mult 2")

$\dim \operatorname{Ker} N = 1 \leftarrow$ only one lin. indep eigenvector

("geom mult. 1")