Math 223, lecture 23

Q: Is every vector an eigenvector of $Id_{\mathcal{U}}$?

A: yes, $Id_{\mathcal{U}} \cdot \underline{u} = 1 \cdot \underline{u} \iff \underline{u}$ is an eigenvector with e.v. 1.

In general, fix $T \in End(\mathcal{U})$. For each λ set

$$\mathcal{U}_\lambda = Ker(T - \lambda \cdot Id_{\mathcal{U}}) = Ker(\lambda - T)$$

if $\mathcal{U}_\lambda \neq \{\underline{0}\}$ call λ an eigenvalue, $\mathcal{U}_\lambda$ the eigenspace (associated to λ), and elements of $\mathcal{U}_\lambda$ eigenvectors.

Call the set of eigenvalues the spectrum of T, write $spec(T)$ for it.

Aside: This notion of "spectrum" is really only for f.d. vector spaces.

Last time: λ is is a eigenvalue iff $\lambda - T$ is singular, iff $\det(\lambda - T) = 0$, iff $p_T(\lambda) = 0$

where $p_T(x) = \det(x Id_{\mathcal{U}} - T)$ is the charcteristic polynomial of T.

Solution of midterm problem 3 using volume forms.

Say $V = W_1 \oplus W_2$ $\quad T = A \oplus B$, $A \in \mathrm{End}(W_1)$, $B \in \mathrm{End}(W_2)$

<u>claim:</u> $\det T = (\det A)(\det B)$

Define maps $\tilde{A}, \tilde{B}$ on V by

$$\tilde{A} = A \oplus \mathrm{Id}_{W_2} \quad , \quad \tilde{B} = \mathrm{Id}_{W_1} \oplus B$$

Then
$$(\tilde{A}\tilde{B})\begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix} = \tilde{A}\begin{pmatrix} \underline{w}_1 \\ B\underline{w}_2 \end{pmatrix} = \begin{pmatrix} A\underline{w}_1 \\ B\underline{w}_2 \end{pmatrix} = T\begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix}$$

so enough to show $\det \tilde{A} = \det A$

Fix a nonzero volume form f on V. $\dim W_1 = n$, $\dim W_2 = m$

consider function

$$g(\underline{u}_1, \dots, \underline{u}_n) = f(\underline{u}_1, \dots, \underline{u}_n, \underline{w}_1, \dots, \underline{w}_m)$$

where $(\underline{u}_1, \dots, \underline{u}_n) \in W_1^n$ are variables,

fixed

$(\underline{w}_1, , \underline{w}_m) \in W_2$ is a basis

g is a volume form on W_1: linear in each $\underline{u}_i$ (f is), zero if $\underline{u}_i = \underline{u}_j$ $(i \neq j)$ (f is)

g is non-zero (if $\{\underline{u}_i\}_{i=1}^{n} \subset W_1$ are a basis, then $\{\underline{u}_i\} \cup \{\underline{w}_j\}$ are a basis of V)

so $\det A \; g(\underline{u}_1, .., \underline{u}_n) =$

$= g(A\underline{u}_1, .., A\underline{u}_n) = f(A\underline{u}_1, , A\underline{u}_n, \underline{w}_1, , \underline{w}_m)$

$= f(\tilde{A}\underline{u}_1, , \tilde{A}\underline{u}_n, \tilde{A}\underline{w}_1, , \tilde{A}\underline{w}_m)$

$= (\det \tilde{A}) \cdot f(\underline{u}_1, , \underline{u}_n, \underline{w}_1, .., , \underline{w}_m)$

$= \det \tilde{A} \cdot g(\underline{u}_1, ., \underline{u}_n)$

But can choose $\{\underline{u}_i\}$ so $g(\underline{u}_1, , \underline{u}_n) \neq 0$

then $\det A = \det \tilde{A}$.

Anomaly: to find eigenvalues it seems you need to know about det.

Not really the case:

Thm: Every linear map on a f.d. vector space has at least one eigenvalue (after extension of scalars)

Pf 1: The char poly $p_T(x)$ is monic, degree $=\dim V$. It has roots

(fact: every complex polynomial has complex roots)

Pf 2: let $\underline{v} \in V$ be a non-zero vector.

consider the vectors $\underline{v} = T^0\underline{v}, T^1\underline{v}, T^2\underline{v}, \ldots, T^n\underline{v}$

That's a sequence of $n+1$ vectors; if we take $n = \dim V$, they must be linearly dependent.

So have scalars $\{a_i\}_{i=0}^{n}$ not all zero, s.t.

$$\sum_{i=0}^{n} a_i(T^i\underline{v}) = \underline{0} \iff \left(\sum_{i=0}^{n} a_i T^i\right)\cdot \underline{v} = 0$$

$$\iff q(T)\cdot \underline{v} = \underline{0} \quad \text{where } q(x) = \sum_{i=0}^{n} a_i x^i.$$

(q is non-constant: if $q = a_0$ then $q_0(T) = a_0$, with $a_0 \neq 0$, then $a_0 \underline{v} = \underline{0}$ would violate $\underline{v} \neq \underline{0}$)

So

Let λ be a <u>root</u> of q. Then $(x-\lambda) \mid q(x)$:

can write $q(x) = (x-\lambda) \cdot r(x)$

Then $\underline{0} = q(T)\,\underline{v} \overset{\text{HW!}}{=} (T-\lambda)\, r(T) \cdot \underline{v} = (T-\lambda)(r(T)\underline{v})$

~~maybe~~ If $r(T)\,\underline{v} \neq \underline{0}$ then it's an eigenvector with ev. λ (killed by $T-\lambda$). But why is $r(T)\underline{v} \neq \underline{0}$?

So let q be the poly of <u>least</u> degree s.t. $q(T)\,\underline{v} = \underline{0}$.

Then $\deg r = \deg q - 1 < \deg q$, so $r(T)\underline{v} \neq \underline{0}$

∎

<u>Aside</u>: ("Power method") If take random $\underline{v}$,

then vectors $\underline{v}, T\underline{v}, T^2\underline{v}, \ldots, T^n\underline{v}, \ldots$

normalised, typically converge to eigenvector corresponding to largest eigenvalue (in magnitude).

(why? suppose $\underline{v} = \sum_{i=1}^{n} a_i \underline{v}_i$, $T\underline{v}_i = \lambda_i$.

then $T^n \underline{v} = \sum_{i=1}^{n} a_i T^n \underline{v}_i = \sum_{i=1}^{n} a_i \lambda_i^n \underline{v}_i$

as $n \to \infty$, the largest λ will dominate)

<u>Example</u>: (complex roots) $A = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$A^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ so if $A\underline{v} = \lambda \underline{v}$,

$$-\underline{v} = A^2 \underline{v} = A(A\underline{v}) = A(\lambda \underline{v}) = \lambda (A\underline{v}) = \lambda^2 \underline{v}$$

so $\lambda^2 = -1$ no real numbers like this

<u>But</u> have complex roots $\pm i$.

(aside $A^2 = -I$ so $A^2 + I = 0$, and charpoly is $p_A(x) = x^2 + 1$).

eigenvectors $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

so $y = \lambda x$, $-x = \lambda y$.

e.g. take $x=1$, get eigenvectors $\begin{pmatrix}1\\i\end{pmatrix}, \begin{pmatrix}1\\-i\end{pmatrix}$

$$\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}i\\-1\end{pmatrix} = i\begin{pmatrix}1\\i\end{pmatrix}$$

$i^2=-1$

$$\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \begin{pmatrix}-i\\-1\end{pmatrix} = -i\begin{pmatrix}1\\-i\end{pmatrix}$$

$(-i)^2=-1$

This will be our approach to extension of scalars: work with matrices, freely switch between real & complex entries

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Trick for homework & exams.

say matrix has integral (or rational) entries (common in examples in Math 223)

Then $P_T(x)$ has integral (or rational) coefficients. Clear denominators to get integral coeff.

<u>Thm</u>: Let $p \in \mathbb{Z}[x]$ have the form

$$p = \sum_{i}^{d} a_i x^i$$

with $a_0 a_d \neq 0$ (i.e. $\deg p = d$, 0 is not a root]

suppose that $p(\frac{r}{s})=0$ where $r,s \in \mathbb{Z}$, $s \neq 0$, $\gcd(r,s)=1$. Then $r \mid a_0$, $s \mid a_d$.

conclusion: to find the rational roots, try all combinations $\frac{r}{s}$, where $r \mid a_0$, $s \mid a_d$

(often $a_d = 1$ so $s=1$)

Pf: Say $p(\frac{r}{s})=0$. Then $\sum_{i=0}^{d} a_i (\frac{r}{s})^i = 0$

multiply by s^d. Get: $\sum_{i=0}^{d} a_i r^i s^{d-i} = 0$

so $\begin{cases} a_0 s^d = -r(\sum_{i=1}^{d} a_i r^{i-1} s^{d-i}) \\ a_d \cdot r^d = -s(\sum_{i=0}^{d-1} a_i r^i s^{d-i-1}) \end{cases}$

so $r \mid a_0 s^d$, $s \mid a_d \cdot r^d$, but r is prime to s,

so $r \mid a_0$, $s \mid a_d$.

Example: Say $p_7(x) = x^2 + 5x + 6$.

~~thm says~~ Rational root thm says: if rational roots exist, must be integers, one of

$\pm 1, \pm 2, \pm 3, \pm 6$ (divisors of 6)

$p(1) = 12$, $p(-1) = 2$, $p(2) = 20$, $\boxed{p(-2) = 0}$

and $\frac{x^2+5x+6}{x+2} = x+3$ so other root is -3.

(beyond 223)

General view: Say V is an F-vsp, $K \supset F$ is a bigger field.

Fix a basis $\{\underline{v}_i\}_{i \in I} \subset V$.

"Define" $V_K = \{ \sum_{i \in I} a_i \underline{v}_i \mid a_i \in K \}$

(check: if change basis, we get an "equivalent" space)

$(\mathbb{R}^n)_{\mathbb{C}} = \{ \sum_{i=1}^{n} a_i \underline{e}_i \mid a_i \in \mathbb{C} \}$, $\underline{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

$\cong \mathbb{C}^n$

$\dim_K V_K = \dim_F V$ (have same basis)

$\dim_F V_K = (\dim_F K) \cdot \dim_F V$

Think of $\mathbb{C}^n$ as a real vsp, has basis:

$\{\underline{e}_j\}_{j=1}^n \cup \{i\underline{e}_j\}_{j=1}^n$