Math 223, lecture 24

Have vsp V, map $T \in \mathrm{End}(V)$, call λ an eigenvalue of T if we have $\underline{v} \in V$, $\underline{v} \neq \underline{0}$ s.t.

$$T\underline{v} = \lambda \underline{v}$$

$$\iff \quad V_\lambda \overset{\text{def}}{=} \mathrm{Ker}(T-\lambda) = \mathrm{Ker}(\lambda - T) \neq \{\underline{0}\}$$

Last time: (1) can detect eigenvalues using polynomials, eg. characteristic polynomial

$$p_T(x) = \det(x\,\mathrm{Id}_V - T).$$

Today: investigate further, look into bases of eigenvectors

Lemma: let $\{\underline{v}_i\}_{i=1}^r \subset V$, $\underline{v}_i$ are (non-zero) eigenvectors corresponding to distinct eigenvalues λ_i.

$$(\text{i.e. } T\underline{v}_i = \lambda_i \underline{v}_i, \quad \lambda_i \neq \lambda_j \text{ if } i \neq j)$$

Then $\{\underline{v}_i\}_{i=1}^r$ are linearly independent.

Pf: let $\sum a_i \underline{v}_i = \underline{0}$ be a minimal dependence. PS 2: this implies all $a_i \neq 0$, and that except for rescaling, this is the only dependence involving the same vectors

But, applying T, we have

$$\underline{0} = T\underline{0} = T\left(\sum_i a_i \underline{v}_i\right) = \sum_i a_i T\underline{v}_i$$

$$= \sum_i (a_i \lambda_i)\, \underline{v}_i.$$

(another dependence with same $\{\underline{v}_i\}$)

But by assumption, λ_i are distinct! so sum must involve one summand

But it can't since all $\underline{v}_i \neq \underline{0}$. This is a contradiction, so the set is independent.

Corollary: The eigenspaces V_λ are independent:

If $B_\lambda \subset V_\lambda$ is a basis then $\bigcup_{\lambda \in \mathrm{Spec}(T)} B_\lambda$ is linearly independent.

Pf: Say have $\sum a_i \underline{v}_i = \underline{0}$ where $\underline{v}_i$

chosen from a B_λ. combining summands in each basis, get

$$\sum_\lambda v_\lambda = 0$$

$$v_\lambda = \sum_{j:\, v_j \in B_\lambda} a_j v_j \in V_\lambda$$

But v_λ are indep if non-zero (lemma) so all are zero.

Then all $a_j = 0$ since B_λ is a basis.

<u>Corollary</u>: $\sum_{\lambda \in \mathrm{Spec}(T)} \dim V_\lambda \leq \dim V$.

<u>Corollary</u>: If $\dim V = n$, T has at most n distinct eigenvalues ($\dim V_\lambda \geq 1$)

<u>Def</u>: The <u>algebraic</u> multiplicity of λ is the largest r s.t. $(x-\lambda)^r \mid p_T$, i.e. its mult. as a root of p_T.

The <u>geometric multiplicity</u> is $\dim V_\lambda$

<u>Q</u>: why is V_λ a subspace?

$V_\lambda = \{ \underline{v} \mid T\underline{v} = \lambda \underline{v} \}$: $T\underline{v} = \lambda \underline{v} \iff (\lambda - T)\underline{v} = \underline{0}$

Pf: $V_\lambda = \ker(\lambda - T)$

$\left(\text{if } T\underline{a} = \lambda \underline{a},\ T\underline{b} = \lambda \underline{b},\ T(\underline{a}+\underline{b}) = T\underline{a} + T\underline{b} = \lambda \underline{a} + \lambda \underline{b} = \lambda(\underline{a}+\underline{b}) \right)$

Prop: Alg. mult $\geq$ Geom mult.

Pf: Let $\{\underline{v}_i\}_{i=1}^{r} \subset V_\lambda$ be a basis.

(r = geom multiplicity)

Extend this to a basis $\{\underline{v}_i\}_{i=1}^{n} \subset V$.

What is the matrix of T in this basis?

$$\left(\begin{array}{cccc|c} \lambda & & & 0 & \\ & \lambda & & & * \\ & & \ddots & & \\ & & & \lambda & \\ \hline & & 0 & & A \end{array} \right)$$

first r columns — last n−r columns

first column is $\begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ since $T\underline{v}_1 = \lambda \underline{v}_1$

second is $\begin{pmatrix} 0 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ since $T\underline{v}_2 = \lambda \underline{v}_2$, etc

A = bottom right $(n-r) \times (n-r)$

$(n-r)\times(n-r)$ square

so matrix of $X - T$ is:

$$\left(\begin{array}{ccc|c} x-\lambda & & 0 & \\ & x-\lambda & & * \\ & & \ddots \quad x-\lambda & \\ \hline & 0 & & x-A \end{array}\right)$$

$$\det(x-T) \underset{\substack{\uparrow \\ \text{minor} \\ \text{expansion by } 1^{st} \text{ col}}}{=} (x-\lambda)\cdot\det\left(\begin{array}{c|c} \underbrace{\begin{matrix} x-\lambda & & \\ & \ddots & \\ & & x-\lambda \end{matrix}}_{r-1} & \underbrace{*}_{n-r} \end{array}\right)$$

$$\underset{\substack{\uparrow \\ \text{continue} \\ r-1 \text{ times}}}{=} (x-\lambda)^r \det(x-A)$$

so $p_T(x) = (x-\lambda)^r \cdot p_A(x)$

so mult of λ as a root is <u>at least</u> r.

<u>Examples:</u> (1) $\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$ has 3 eigenvalues

each has alg. mult. $1 \Rightarrow$ geom mult 1.

(2)
$\mathcal{U} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $\quad \mathcal{U}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

so $\mathcal{U}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \lambda = 1$, then need $x+y=y$ so $y=0$.

So $(\mathbb{R}^2)_1 = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

But the char poly is $\det\begin{pmatrix} x-1 & -1 \\ 0 & x-1 \end{pmatrix} = (x-1)^2$

(matrix is upper-triangular).

so alg. mult. is 2.

(3) <u>Example:</u> $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}$

$$P_A(x) = \begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ 0 & 2 & x+1 \end{vmatrix} \underset{\uparrow}{=} x \begin{vmatrix} x & -1 \\ 2 & x+1 \end{vmatrix} =$$

expand by 1st col

$$= x\left(x(x+1)+2\right) = x(x^2+x+2) = x^3+x^2+2x$$

(rational root thm: rational roots

are 0, possibly also $\pm 1, \pm 2$, $\left((-1)^2 + 2 - 1 = 2 \right.$
$\left. (-2)^2 - 2 + 2 = 4 \right)$

roots are $0, \frac{-1 \pm \sqrt{-7}}{2}$.

So A has 3 distinct eigenvalues
$\Rightarrow$ each has mult. 1.

<u>Def:</u> Call T diagonable (or diagonalizable)

if V has a basis consisting of eigenvalues of T.

$\Leftrightarrow$ if some matrix of T is diagonal

$\Leftrightarrow$ if the matrix of T is similar to a diagonal matrix.

Recalling our theory of change-of-basis,

let A be a $\overset{n \times n}{\text{square}}$ matrix, suppose

$\{\underline{v}_i\}_{i=1}^n$ are $\overset{\text{linearly indep}}{\text{eigenvectors}}$ of A: $T\underline{v}_i = \lambda_i \underline{v}_i$

Let S = matrix whose columns are the $\underline{v}_i$:

Then $S A S = D$, where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$\iff A = SDS^{-1}$.

<u>Recall example:</u> $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ $p_A(x) = \begin{vmatrix} x-3 & -4 \\ -2 & x-3 \end{vmatrix}$

$= (x-3)(x-3) - 8 = x^2 - 6x + 1$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{32}}{2} = 3 \pm \sqrt{8}.$$

eigenvectors $\begin{aligned} 3x + 4y &= \lambda x \\ 2x + 3y &= \lambda y \end{aligned}$ $\iff \frac{(3-\lambda)}{4} x = -y$

ev. $\begin{pmatrix} 1 \\ \frac{\lambda - 3}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\pm\sqrt{8}}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}$.

so $S = \begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$S^{-1} A S = \begin{pmatrix} 3 + 2\sqrt{2} & 0 \\ 0 & 3 - 2\sqrt{2} \end{pmatrix}$$

$$A = S \begin{pmatrix} 3 + 2\sqrt{2} & 0 \\ 0 & 3 - 2\sqrt{2} \end{pmatrix} S^{-1}$$

where $S = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

(here mult is 1)

Remarks: ① If A is "random", it has n distinct eigenvalues

② Can define "generalized eigenspaces" s.t. $V = \bigoplus_{\lambda} V_\lambda$. ← generalized eigenspaces

$\dim\left(\begin{smallmatrix}\text{gen.}\\ \text{eigenspace}\end{smallmatrix}\right) = $ alg. multiplicity.

(see "Jordan canonical form", e.g. my notes for MATH 412)

E.g. $W = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ is diagonable.

$$W\begin{pmatrix}1\\ \pm i\end{pmatrix} = \pm i\begin{pmatrix}1\\ \pm i\end{pmatrix}$$

so W is diagonable: $S = \begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$

$$S^{-1}WS = \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}.$$