Math 223, lecture 25

Q: recursion relations

A recursion relation is a sequence defined by $a_n = F(a_0, \dots, a_{n-1})$

especially $a_n = F(a_{n-1}, \dots, a_{n-k})$

often occur in combinatorics, eg. the Pingala-Fibonacci sequence, defined by

$$F_{n+1} = F_n + F_{n-1}$$

(Ex: Enumerate words in alphabet 0, 1, without 11, 000, start with 0, end with 1:

i.e. such a word is a sequence of pieces of form 01, 001.

Let a_n be the number of words of length n. last piece is either 01 or 001., so

$$a_n = a_{n-2} + a_{n-3}$$

counts words of length $n-2$ + 01   counts words of length $n-3$ + 001

+ $a_0 = 1$, $a_1 = 0$, $a_2 = 1$. (initial condition)

HW: general method for solving such recurrence relations

("constant coeff linear recurrences").

Last time: multiplicity.

$\dim V = n$, $T \in \mathrm{End}(V)$, p_T = char poly,

$V_\lambda = \ker(\lambda - T)$

Def: Let λ be a root of $p_T \Leftrightarrow$ an eigenvalue of T. Then

$\dim V_\lambda$ = geometric multiplicity of λ

$\mathrm{ord}_\lambda p_T$ = algebraic multiplicity of λ

Prop: ~~[illegible]~~ If $r = \dim V_\lambda$, then $(x-\lambda)^r \mid p_T$.

Best (generic) case: all multiplicities are 1.

Then T is diagonable.

(not neccessary, eg. I_n is diagonable)

Usually, if mult. >1, have an explanation: a symmetry in problem

Suppose that $S \in End(V)$ commutes with T $ST = TS$.

HW: S acts on each V_λ : if $\underline{v} \in V_\lambda$ so is $S\underline{v}$

Eg. (Physics) Angular momentum:

V = "reasonable" functions on $\mathbb{R}^3$.

$$T = -\underbrace{\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)}_{\Delta} + V\left(\sqrt{x^2+y^2+z^2}\right)$$

Ex: for any rotation R on $\mathbb{R}^3$ (acting on functions) $RT = TR$.

General picture:

(1) $\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$ diagonalable, distinct eigenvalues

(2) $\begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ " , geom. mult > 1

(3) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ non-diagonalable: alg. mult > geom mult.

<u>Another example</u>: $T = -\frac{d^2}{dx^2}$ on functions on circle.

<u>Saw</u>: $\sin(2\pi kx)$, $\cos(2\pi kx)$ are eigenfunctions (= eigenvectors), $k \geq 0$

$$T(\sin(2\pi kx)) = 4\pi^2 k^2 \sin(2\pi kx)$$

$$T(\cos(2\pi kx)) = 4\pi^2 k^2 \cos(2\pi kx)$$

$\Rightarrow$ multiplicity 2 (except when $k=0$)

hidden symmetry: use complex exponentials

$\{e^{2\pi ikx}\}_{k \in \mathbb{Z}}$ $\quad$ $\left(e^{i\theta} = \cos\theta + i\sin\theta \right.$

eigenfunctions of $\frac{d}{dx}$. $\left(e^z = \sum_{n=0} \frac{1}{n!} z^n \right)$

$$\frac{d}{dx}\left(e^{2\pi i k x}\right) = (2\pi i k) e^{2\pi i k x}$$

eigenvalues distinct.

$T = \left(i\frac{d}{dx}\right)^2$, so eigenvalues of T are $-\lambda^2$, $\lambda =$ eigenvalue of $\frac{d}{dx}$.

in particular $-\lambda^2$ eigenspace of T = union of eigen spaces corresp to $\pm\lambda$.

Or:

T commutes with reflection:

$$(Pf)(x) = f(-x)$$

so eigenspaces of T can contain both even and odd functions.

Diagonalization: Move to a basis where the map has a diagonal matrix

$\left(\text{e.g. } V = \left\{ \begin{array}{l} \text{functions} \\ \text{on } \mathbb{R} \text{ s.t.} \\ f(x+1) = f(x) \end{array} \right\} \right)$, $T = \frac{d}{dx}$,

T

basis $\{e^{2\pi i k x}\}_{k\in\mathbb{Z}}$.

Eg.: $V = \mathbb{R}[x]^{\le 3}$, $T = x\frac{d}{dx} = M_x \cdot D$

$T(x) = M_x \cdot D \cdot x = M_x \cdot 1 = x = 1\cdot x$

$T(x^k) = M_x(Dx^k) = M_x \; k\cdot x^{k-1} = k\, x^k$.

in basis $\{x^0, x^1, x^2, x^3\}$ matrix of T is

$\mathrm{diag}(0, 1, 2, 3)$.

What about $T = M_{x+1} \cdot D$.

in basis $\{1, x, x^2, x^3\}$, matrix is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ not diagonal

But $M_{x+1} D((x+1)^k) = k\,(x+1)^k$

so in basis $\{(x+1)^k\}_{k\le 3}$ matrix is

$\mathrm{diag}(0, 1, 2, 3)$

Different pov: char poly of B =

$$\begin{vmatrix} x-0 & & * & \\ & x-1 & & \\ & & x-2 & \\ 0 & & & x-3 \end{vmatrix} = (x-0)(x-1)(x-2)(x-3)$$

dont care about these

so matrix of T triangular $\Rightarrow$ can read eigenvalues off the diagonal.

Eg. $T = x\frac{d}{dx} + \frac{d^2}{dx^2} = M_x D + D^2$.

$T(x^k) = k x^k + x^{k-2}(k(k-1))$

matrix is upper-triangular:

$$\begin{pmatrix} 0 & 0 & 2 & 0 & \cdots & 0 & \\ & 1 & 0 & 6 & \cdots & 0 & \\ & & 2 & 0 & 12 & \vdots & \\ & & & 3 & 0 & k(k-1) & \\ & & & & 4 & 0 & \\ & 0 & & & & k & \\ & & & & & & \ddots \end{pmatrix}$$

eigenvalues still $0, 1, 2, \ldots, k, \ldots$

$T \cdot 1 = 0, \quad T \cdot x = x. \quad Tx^2 = 2x^2 + 2$

want a, b s.t. $T(x^2 + ax + b) = 2(x^2 + ax + b)$

i.e. $2x^2 + 2 + a = 2x^2 + 2ax + 2b$

so take $a = 0$, $b = 1$

$T(x^2+1) = 2x^2+2 = 2(x^2+1)$

Want $T(x^3+ax^2+bx+c) = 3(x^3+ax^2+bx+c)$

$\quad \| $

$3x^3 + 6x + 2ax^2 + 2a + b$

want $2ax^2 = 3ax^2$, $6x = 3bx$,
$2a+b = 3c$

so want $a=0$, $b=2$, $c = 2/3$.

Take $T(x^3+2x+4/3) = 3x^3+6x+2$

$= 3(x^3+2x+4/3)$

In basis $\{1, x, x^2+1, x^3+2x+4/3\}$

matrix is $\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$.

<u>Ex:</u> modify x^3 by

$$x^3 + a(x^2+1) + bx + c$$

previous eigenvectors we discovered

i.e.: we expect p_3 = eigenvector starting with x^3. Then have $x^3 = p_3 + ap_2 + bp_1 + cp_0$

$$p_2 = x^2+1, \quad p_1 = x, \quad p_0 = 1$$

apply T, set $Tx^3 = 3x^3 + 6x$

$$= 3(p_3 + ap_2 + bp_1 + cp_0) + 6p_1$$

but $T(\qquad) = 3p_3 + 2ap_2 + 1bp_1 + 0p_0$

In general, if I know λ, finding V_λ amounts to solving linear equation

$$(T - \lambda)\underline{v} = \underline{0}$$

$$\left[\begin{pmatrix} & 0 & 2 & 0 \\ & 1 & 0 & 6 \\ & & 2 & 0 \\ & & & 3 \end{pmatrix} - \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}\right]\underline{v} = \underline{0}$$

ise the system

$$\left[\begin{array}{cccc|c} -2 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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functions of two variables

$$\Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2}$$

$$(R(\alpha)\cdot f)(r,\theta) = f(r, \theta-\alpha)$$

$$R(\alpha)\cdot f)\begin{pmatrix} x \\ y \end{pmatrix} = f\left(R(-\alpha)\cdot\begin{pmatrix} x \\ y \end{pmatrix}\right) \quad R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$\Delta \cdot R(\alpha) = R(\alpha) \cdot f$$

so instead of studying Δ on all functions,

study Δ on each eigenspace of rotations

$$R(\alpha)f = kf \iff f(r, \theta-\alpha) = k\, f(\theta)$$

$$\iff f(r,\theta) = g(r)\cdot e^{k\theta}$$

amika?

so let $V_k = \{ g(r) \cdot e^{2\pi i k\theta} \}$

If $f(r,\theta) \in V_k$, $R(\alpha) f = e^{-2\pi i k \alpha} f$

study Δ on each V_k (

On V_k, $\Delta\left(g(r) e^{2\pi i k\theta}\right) =$

$$= \left(g'' + \frac{1}{r} g' - \frac{4\pi^2 k^2}{r^2} g\right) e^{2\pi i k\theta}$$

so $\Delta f = \lambda f \iff$

$$\boxed{g'' + \frac{1}{r} g' - \frac{4\pi^2 k^2}{r^2} g = \lambda g}$$

ODE instead of PDE (linear!)