Math 223, lecture 26

<u>Q:</u> let T be diagonable, with matrix $\mathrm{diag}(\lambda_1, .., \lambda_n)$ in basis of eigenvectors.

is $\det T = \prod_{i=1}^{n} \lambda_i$?

<u>A:</u> yes: $\det T = \det\begin{pmatrix}\text{matrix} \\ \text{of } T\end{pmatrix}$

(similar matrices have same determinant:

$$\det(SAS^{-1}) = \cancel{(\det S)}(\det A)\cancel{(\det S)}^{-1}$$
$$= \det A.$$

<u>Also:</u> (HW) char poly of T is

$$p_T(x) = \det(x - T) = \prod_{i=1}^{n}(x - \lambda_i)$$

plug in $x=0$. (det = prod of e.v. counted with <u>alg.</u> multiplicity)

<u>Setup:</u> V f.d. vsp, $T \in \mathrm{End}(V)$

say that T is diagonable (diagonalizable) if have a basis of V consisting of eigenvectors of T.

$\iff$ if T has matrix which is diagonal.

<u>Applications:</u> easier to work with diagonal matrices

<u>Example:</u> let A be a square matrix, $A \in M_n(\mathbb{R})$

<u>Def:</u> The <u>trace</u> of A is the number

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}.$$

<u>Lemma:</u> if A, B are similar then $\operatorname{tr} A = \operatorname{tr} B$

<u>Pf:</u> Say $B = SAS^{-1}$. Then

$$\operatorname{Tr} B \stackrel{\text{def}}{=} \sum_{i=1}^{n} B_{ii} = \sum_{i} \left(\sum_{j,k} S_{ij} A_{jk} S^{-1}_{ki} \right)$$

$$= \sum_{i,j,k} S_{ij} A_{jk} S^{-1}_{ki}$$

$$= \sum_{j,k} A_{jk} \left(\sum_{j} S^{-1}_{ki} S_{ij} \right)$$

$$= \sum A_{ik} \cdot (S^{-1}S)_{1} \quad \leftarrow S^{-1}S = I_n$$

$$= \sum_{j,k} A_{jk} \cdot \delta_{jk}$$

$$= \sum_{j} A_{jj} = \operatorname{Tr} A$$

$(S^{-1}S)_{kj} = \begin{cases} 1 & k=j \\ 0 & k\neq j \end{cases} = \delta_{kj}$

<u>Cor:</u> Define for $T \in \operatorname{End}(V)$, $\operatorname{Tr} T = \operatorname{Tr} A$ for any matrix of A.

<u>Prop:</u> Say T is diagonalble with eigenvalues $\{\lambda_i\}_{i=1}^n$. Then
$$\operatorname{Tr} T = \sum_{i=1}^n \lambda_i.$$

<u>Pf:</u> One matrix of T is $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$

Again, if T is any matrix, $p_T(x) = \prod_{i=1}^n (x - \lambda_i)$

then $\operatorname{Tr} T = \sum_{i=1}^n \lambda_i$.

"<u>Proofs</u>": (1) $\operatorname{Tr} T$ is continuous in T, $\sum_{i=1}^n \lambda_i$ is cts (λ_i are cts in T)

the two are equal on the the dense set of diagonable matrices

(2) Look at $\det(x - A)$:

$$\det\begin{pmatrix} x-a_{11} & & & * \\ & x-a_{22} & & \\ & & \ddots & \\ * & & & x-a_{nn} \end{pmatrix} = \text{poly in } x.$$

$$\| $$

$$\sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=1}^{n} (x-A)_{i,\sigma(i)}$$

$\uparrow$ bijections $[n]\to[n]$

choose n entries from matrix, all from different rows, cols, multiply + sign, add all possibilities

coefficient of x^{n-1} = ?

to get ~~xe~~ x^{n-1} must choose $n-1$ of the diagonal entries. The only row + col missing an entry are of the other diagonal entry.

So, coeff of x^{n-1} in $p_T(x)$

$$= \text{coeff of } x^{n-1} \text{ is } (x-a_{11})\cdots(x-a_{nn})$$

Example:

$$\begin{pmatrix} x-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & x-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & x-a_{33} \end{pmatrix}$$

(1^{st} entry of 1^{st} row, 3^{rd} " " 3^{rd} row, in 2^{nd} row must choose 2^{nd} entry)

$\Rightarrow$ coeff of x^{n-1} in $\prod_{i=1}^{n}(x-\lambda_i)$, $\prod_{i=1}^{n}(x-a_{ii})$ are equal. These are $-\sum_{i=1}^{n}\lambda_i$, $-\sum_{i=1}^{n}a_{ii}$ respectively. So $\sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n}a_{ii}$.

Aside: Also true: every matrix is similar to an upper-triangular matrix.
(Proves the claims too)

Summary: An endomorphism $T \in \mathrm{End}(V)$ ($\dim V = n$) has a trace $\mathrm{Tr}\, T$, has a determinant $\det T$.

We have $\mathrm{tr}\, T = \sum_{i=1}^{n}\lambda_i$, $\det T = \prod_{i=1}^{n}\lambda_i$ where we count eigenvalues with alg. multiplicity.

Proof easy for diagonable maps, uses characteristic poly in general

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~~If~~ $T \in End(V)$, what do we call V_0?
A: it's the <u>kernel</u> $\ker(T)$

So nullity of T = geom multiplicity of the eigenvalue 0.

<u>Applications</u>: (1) Power method
(2) random walks
(3) Google Page Rank

Say T is diagonable How do we find an eigenvector?

<u>Idea</u>: "power method". Define $\|\underline{v}\|_\infty = \max_i |v_i|$ if $\underline{V} = \mathbb{R}^n$

<u>check</u>: $\|a\underline{v}\| = |a| \cdot \|\underline{v}\|$, $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\|_\infty + \|\underline{v}\|_\infty$

Say $\{\underline{v}_i\}_{i=1}^n \subseteq \mathbb{R}^n$ is a basis of eigenvectors,

$\{\lambda_i\}_{i=1}^n \subset \mathbb{C}$ are the eigenvalues, say $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \geq \cdots$

spectral gap

Take random vector $\underline{v}$. We have $\underline{v} = \sum_{i=1}^{n} a_i \underline{v}_i$ for some a_i. Generically All $a_i \neq 0$

Let k be large, consider $A^k \underline{v}$

$$A^k \underline{v} \underset{\text{linearity}}{=} \sum_{i=1}^{n} a_i A^k \underline{v}_i = \sum_{i=1}^{n} a_i \lambda_i^k \underline{v}_i$$

$$= \lambda_1^k \left(a_1 \underline{v}_1 + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1} \right)^k a_i \underline{v}_i \right)$$

As k grows, all $\underline{v}_i$ are suppressed except for $\underline{v}_1$

Actually, rescale vectors to have size 1.

Set $\underline{w}^{(0)} = \underline{v}$, $\underline{w}^{(k+1)} = \dfrac{A \cdot \underline{w}^{(k)}}{\| A \underline{w}^{(k)} \|_\infty}$

($\dfrac{\text{vector}}{\text{length}}$ = vector of length 1)

(aside: because $\| a \underline{v} \| = |a| \, \| \underline{v} \|$, $\underline{w}^{(k)} = \dfrac{A^k \underline{v}}{\| A^k \underline{v} \|_\infty}$)

(computationally, rescale at every step.)

(theoretically can rescale at end)

rescaled vectors $\underline{w}^{(k)} \to$ multiple of $\underline{v}_1$

if I have a good approximation to $\underline{v}_1$, can find λ_1 by applying A.

<u>side remark:</u> only used multiplication by A

very efficient if A is sparse (mostly 0)

<u>Aside:</u> can use power method to find r largest eigenvalues: take r vectors

$$\underline{w}_1^{(k)}, \underline{w}_r^{(k)}, \text{ apply } A \text{ to get}$$

$$A\underline{w}_1^{(k)}, \ , A\underline{w}_r^{(k)}$$

remove $\underline{w}_1^{(k+1)}$-component from $\underline{w}_2^{(k+1)}$,

$\underline{w}_1, \underline{w}_2$ - " " $\underline{w}_3$

$\vdots$

(works best if A is symmetric)

Observe: ("inverse iteration") eigenvalues of $(A-\lambda)^{-1}$ are $\frac{1}{\lambda_i-\lambda}$.

So largest eigenvalue of $(A-\lambda)^{-1}$ corresponds to eigenvalue of A closest to λ

(now define $\underline{w}^{(k+1)}$ as solution to eqn

$$(A-\lambda)\,\underline{w}^{(k+1)} = \underline{w}^{(k)}.$$

Netflix problem: have a large matrix

$N \in M_{I \times J}(\mathbb{R})$ $\quad I$ = users
J = movies

problem: we know a few of the entries, want to fill in the rest.

one assumption: $\operatorname{rk} N$ is small

If A is symmetric, have algorithms for finding all eigenvalues & eigenvectors. More expensive.

Book on numerical eigenvalues (free, online)

Bai - Demmel - Dongarra - Ruhe - van der Vorst

"templates for the solution of algebraic eigenvalue problems"