Math 223, Lecture 28

Recently: Eigenvectors, Eigenvalues, Diagonalization.

Today: Start last part of course: inner product spaces.

plans: (1) motivation (2) definition (3) examples

next week: connect inner products to diagonalization.

In this part of the course, the field of scalars in $\mathbb{R}$ or $\mathbb{C}$. (rest of the course made sense over any field)

Our vector spaces so far were "floppy": no notion of distance or angle between vectors

In many situations, starting with Euclidean span $\mathbb{E}^n$, we want such a notion (or have one)

We will define a notion of "vector space with angle & distance information".

<u>useful</u>: (1) to calculate, in a vsp which has this structure
(2) to model (Euclidean) geometry.

<u>Definition</u>: Let V be a real vsp. An <u>inner product</u> on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ s.t.

(1) ("Bilinearity") $\langle \underline{u}, \alpha \underline{v} + \beta \underline{w} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle + \beta \langle \underline{u}, \underline{w} \rangle$

(2) ("Symmetry"): $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$

(3) ("Positivity") $\langle \underline{u}, \underline{u} \rangle \geq 0$, equality iff $\underline{u} = \underline{0}$

$\mathbb{R}$ enters here

An <u>inner product space</u> is a pair $(V, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is an inner product on V.

<u>Example</u>: $V = \mathbb{R}^n$, <u>standard inner product</u> ("dot product")

$$\langle \underline{u}, \underline{v} \rangle = \sum_{i=1}^{n} u_i v_i$$

((1), (2) clear, and $\langle \underline{u}, \underline{u} \rangle = \sum_{i=1}^{n} u_i^2 \geq 0$)

(over $\mathbb{C}$ this won't work: u_i^2 can be anything)

Def: Let V be a complex vsp. A hermitian product on V is a function $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ s.t.

(1) (Conjugate linearity)

$$\langle \underline{u}, \alpha\underline{v}+\beta\underline{w}\rangle = \alpha\langle \underline{u},\underline{v}\rangle + \beta\langle \underline{u},\underline{w}\rangle$$

(2) ("conjugate symmetry")

$$\langle \underline{u},\underline{v}\rangle = \overline{\langle \underline{v},\underline{u}\rangle}$$

complex conjugate

(3) (Positivity) $\langle \underline{u},\underline{u}\rangle \geq 0$ with equality iff $\underline{u}=\underline{0}$.

A hermitian space is a pair $(V,\langle\cdot,\cdot\rangle)$ with V, $\langle\cdot,\cdot\rangle$ as above.

Since $\overline{z}=z$ if $z\in\mathbb{R}$, we can use 2^{nd} defn in both cases

Lemma: Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space or a Hermitian space. Then:

(1) $\langle \alpha\underline{u}+\beta\underline{v}, \underline{w}\rangle = \overline{\alpha}\langle \underline{u},\underline{v}\rangle + \overline{\beta}\langle \underline{v},\underline{w}\rangle$

(2) $\langle \underline{u},\underline{u}\rangle \in \mathbb{R}$ for all $\underline{u}$

symmetry

Pf: (1) $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle \overset{*}{=} \overline{\langle \underline{w}, \alpha \underline{u} + \beta \underline{v} \rangle}$

$$= \overline{\alpha \langle \underline{w}, \underline{u} \rangle + \beta \langle \underline{w}, \underline{v} \rangle}$$

linearity in 2^{nd} var

$$= \bar{\alpha}\, \overline{\langle \underline{w}, \underline{u} \rangle} + \bar{\beta}\, \overline{\langle \underline{w}, \underline{v} \rangle}$$

$$= \bar{\alpha} \langle \underline{u}, \underline{w} \rangle + \bar{\beta} \langle \underline{v}, \underline{w} \rangle .$$

symmetry

(2) $\langle \underline{u}, \underline{u} \rangle = \overline{\langle \underline{u}, \underline{u} \rangle}$ by symmetry.

$\blacksquare$

Example: The standard inner product on $\mathbb{C}^n$ is

$$\langle \underline{u}, \underline{v} \rangle = \sum_{i=1}^{n} \overline{u_i} \cdot v_i . \qquad \left(\overline{\overline{u_i} v_i} = \overline{v_i} u_i \right)$$

Warning: many mathematician define hermitian products to be linear in 1^{st} argument, conjugate linear in 2^{nd}.

Example: $V = C(a,b)$ = cts functions on $[a,b]$ (real- or complex-valued)

$$\langle f, g \rangle = \int_a^b \overline{f(x)}\, g(x)\, dx$$

Linearity is clear from properties of $\int_a^b$

For $f = g$, $\langle f, f\rangle = \int_a^b |f|^2 dx$,

If $f(x_0) \neq 0$ then $|f(x_0)|^2 > 0$, and this holds on an interval near x_0 (continuity of f) so $\int_a^b |f|^2 dx > 0$.

use this term for both notions

<u>Lemma</u>: Let $(V, \langle\cdot,\cdot\rangle)$ be an inner prdt space, & let $W \subset V$ be a subspace. Then $(W, \langle\cdot,\cdot\rangle|_{W\times W})$ is an inner prdt space.

(2) If V is complex, $(V, \mathrm{Re}\langle\cdot,\cdot\rangle)$ is a real inner prdt space if we think of V as a real vsp.

<u>Pf</u>: Ex. (all axioms are universal)

<u>Example</u>: (Question from start of lecture)

Let V be a vsp, $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$ be two inner products on V.

Let $t, s \geq 0$, not both zero

Then $\langle \underline{u}, \underline{v} \rangle = t\langle \underline{u}, \underline{v} \rangle_1 + s\langle \underline{u}, \underline{v} \rangle_2$
is also an inner product.

Pf: ~~Ex~~. check axioms (1), (2) immediately.

$\langle \underline{u}, \underline{u} \rangle = t\langle \underline{u}, \underline{u} \rangle_1 + s\langle \underline{u}, \underline{u} \rangle_2 \geq 0$

all 4 numbers are ≥ 0

& equality $\Rightarrow$ at least one of $\langle \underline{u}, \underline{u} \rangle_1, \langle \underline{u}, \underline{u} \rangle_2$ to vanish so $\underline{u} = \underline{0}$. ∎

(Aside: the space of inner products on V has very interesting curved geometry)

Two remarks: (1) often use weight functions

$$\langle f, g \rangle = \int_\Omega \overline{f(x)} g(x)\, w(x)\, dx$$

($w(x) > 0$ on Ω) e.g. on $\mathbb{R}[x]$ define

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x)\, e^{-x^2} dx.$$

(2) linear maps can interact with the inner product.

$V = \{$ diff functions on $[a,b]$ with $\underbrace{f(a)=f(b)}_{=0}\}$

$Df = \frac{d}{dx}$. We have: $\int_a^b \overline{\frac{df}{dx}}\, g(x) = -\int_a^b \bar{f} \frac{dg}{dx} dx$

$\Leftrightarrow \quad \langle Df, g \rangle = -\langle f, Dg \rangle$

(Physicists prefer the operator $\hat{p} = iD$

[really $-i\hbar \frac{d}{dx}$, $\hbar$ = constant].

<u>Now</u>: $\langle \hat{p} f, g \rangle = \langle -i \frac{df}{dx}, g \rangle =$

$= i \langle \frac{df}{dx}, g \rangle = -i \langle f, \frac{dg}{dx} \rangle$

conjugate linearity — int. by parts

$= \langle f, -i\frac{dg}{dx} \rangle = \langle f, \hat{p} g \rangle$

linearity — def of $\hat{p}$.

so $\langle \hat{p} f, g \rangle = \langle f, \hat{p} g \rangle$.

(say "$\hat{p}$ is symmetric wrt $\langle \cdot, \cdot \rangle$")

<u>Monday</u>: geometry <u>Wed</u>: bases in this world

On space of functions, $f^2(x)g^2(x)$ is not linear in f, g.

$f(x_0)\, g(x_0)$

this is not "positive": $|f(x_0)|^2 = 0$ does not mean $f = 0$

(this is "positive semidefinite")

(usual inner pdt is "positive definite")

$w(x) = x^2$ on $[-1,1]$ not a problem:

if $\int_{-1}^{1} |f(x)|^2 x^2 \, dx = 0$ then $f = 0$

but if $w(x) = \begin{cases} 1 & |x| \geq \frac{1}{2} \\ 0 & |x| < \frac{1}{2} \end{cases}$

then we really are working on $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$

$\int_{-1}^{1} \bar{f} g \, w \, dx$ defines an inner pdt on

$C([-1,-\frac{1}{2}] \cup [\frac{1}{2},1])$

but only a semidefinite pdt on $C([-1,1])$.