Math 223, lecture 29

Last time: inner products

Field: $\mathbb{R}$ or $\mathbb{C}$, $\overline{a+ib} = a-ib$

Inner pdt: $\langle \cdot, \cdot \rangle : V \times V \to$ field s.t.

(1) $\langle \underline{u}, \alpha \underline{v} + \beta \underline{w} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle + \beta \langle \underline{u}, \underline{w} \rangle$

(2) $\langle \underline{u}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{u} \rangle}$

(3) $\langle \underline{u}, \underline{u} \rangle \geq 0$, and equality iff $\underline{u} = \underline{0}$

Today: distance, angles, Cauchy-Schwartz ineq.

Fix an inner product space V.

Def: The norm of $\underline{u} \in V$ is the real number

$$\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$$

(think of this as the length of $\underline{u}$)

Observe: $\|\underline{u}\|$ well-defined by (3), $\|\underline{u}\| = 0$ iff $\underline{u} = \underline{0}$.

Also, the norm is 1-homogeneous:

$$\|\alpha \underline{u}\| = \sqrt{\langle \alpha \underline{u}, \alpha \underline{u} \rangle} = \sqrt{\bar{\alpha} \cdot \alpha \langle \underline{u}, \underline{u} \rangle}$$

$$= \sqrt{\bar{\alpha}\alpha} \langle \underline{u}, \underline{u} \rangle = |\alpha| \langle \underline{u}, \underline{u} \rangle$$

def of $|z|$

(e.g. if $\alpha = a + bi$, $\bar{\alpha} = a - bi$, $\alpha\bar{\alpha} = a^2 + b^2$

so $|\alpha| = \sqrt{\alpha\bar{\alpha}} = \sqrt{a^2+b^2}$, norm of $\begin{pmatrix} a \\ b \end{pmatrix}$ wrt standard inner pdt on $\mathbb{R}^2$.

$$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

std inner pdt

(Later: $\|\cdot\|$ is a measure of distance, in that

$$\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

)

<u>Lemma:</u> (Cauchy-Schwartz inequality)

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \cdot \|\underline{v}\|$$

(with equality iff $\underline{u}, \underline{v}$ are multiples of each other)

<u>Pf:</u> If $\langle \underline{u}, \underline{v} \rangle = 0$ there is nothing to prove (except for equality)

Otherwise let $\alpha = \dfrac{\langle \underline{u}, \underline{v} \rangle}{|\langle \underline{u}, \underline{v} \rangle|}$, $|\alpha| = \dfrac{|\langle \underline{u}, \underline{v} \rangle|}{|\langle \underline{u}, \underline{v} \rangle|} = 1$

also, $\langle \alpha \underline{u}, \underline{v} \rangle = \bar{\alpha} \cdot \langle \underline{u}, \underline{v} \rangle =$

$$= \frac{\overline{\langle \underline{u}, \underline{v} \rangle}}{|\langle \underline{u}, \underline{v} \rangle|} \cdot \langle \underline{u}, \underline{v} \rangle = \frac{|\langle \underline{u}, \underline{v} \rangle|^2}{|\langle \underline{u}, \underline{v} \rangle|}$$

$= |\langle \underline{u}, \underline{v} \rangle|$.

So replacing $\underline{u}$ with $\alpha \underline{u}$, have $\langle \alpha \underline{u}, \underline{v} \rangle = |\langle \underline{u}, \underline{v} \rangle|$

Now let $f(t) = \| \alpha \underline{u} + t \underline{v} \|^2$ for $t \in \mathbb{R}$.

Then $f(t) = \langle \alpha \underline{u} + t \underline{v}, \alpha \underline{u} + t \underline{v} \rangle =$

$$= t^2 \langle \underline{v}, \underline{v} \rangle + t \langle \alpha \underline{u}, \underline{v} \rangle + t \langle \underline{v}, \alpha \underline{u} \rangle + \langle \alpha \underline{u}, \alpha \underline{u} \rangle$$

choice of α $\longrightarrow$ symmetry

$$= t^2 \langle \underline{v}, \underline{v} \rangle + t\, |\langle \underline{u}, \underline{v} \rangle| + t\, \overline{|\langle \alpha \underline{u}, \underline{v} \rangle|} + |\alpha|^2 \langle \underline{u}, \underline{u} \rangle$$

$$= t^2 \langle \underline{v}, \underline{v} \rangle + 2t\,|\langle \underline{u}, \underline{v} \rangle| + \langle \underline{u}, \underline{u} \rangle$$

$$= \|\underline{v}\|^2 \cdot t^2 + 2|\langle \underline{u}, \underline{v} \rangle| t + \|\underline{u}\|^2.$$

This is a quadratic in $\mathbb{R}[t]$, always non negative ($f(t) = \|\cdot\|^2 \geq 0$)

so we complete the square:

$$f(t) = \left(t \cdot \|\underline{v}\| + \frac{|\langle \underline{u}, \underline{v} \rangle|}{\|\underline{v}\|} \right)^2 + \left(\|\underline{u}\|^2 - \frac{|\langle \underline{u}, \underline{v} \rangle|^2}{\|\underline{v}\|^2} \right)$$

since $f(t) \geq 0$ for all t, have

$$\|\underline{u}\|^2 - \frac{|\langle \underline{u}, \underline{v} \rangle|^2}{\|\underline{v}\|^2} \geq 0$$

so $|\langle \underline{u}, \underline{v} \rangle|^2 \leq \|\underline{u}\|^2 \cdot \|\underline{v}\|^2.$

equality holds iff have t s.t. $f(t) = 0$

iff have t s.t. $t\underline{v} + \alpha \underline{u} = \underline{0}$

$$\iff \underline{v} = -(t^{-1}\alpha)\underline{u}.$$

$\blacksquare$

Many applications:

$$\left|\int_a^b f(x)\, g(x)\, dx\right| \le \left(\int_a^b |f|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2}$$

Prop: (Minkowsky's inequality) $\|\underline{u}+\underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$

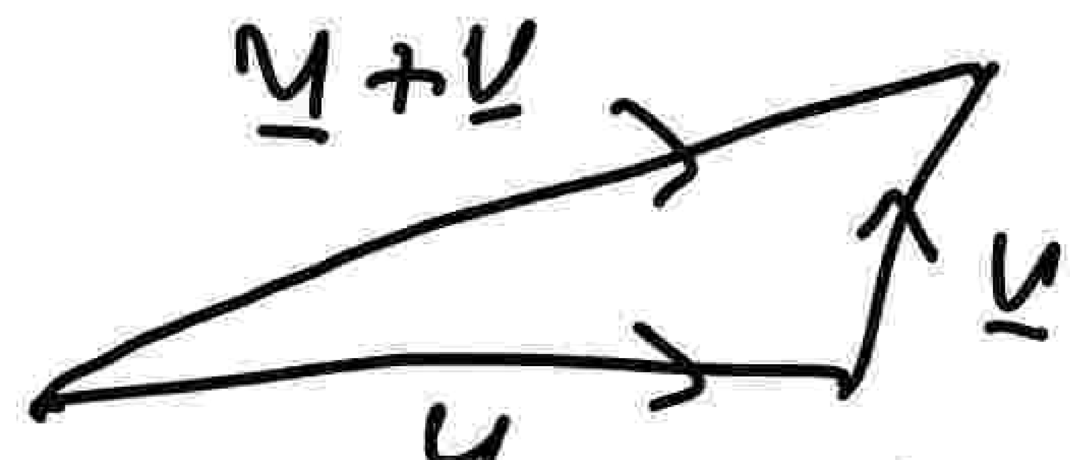
Pf: $\|\underline{u}+\underline{v}\|^2 = \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle$

$$= \langle \underline{u}, \underline{u} \rangle + \langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{u} \rangle + \langle \underline{v}, \underline{v} \rangle$$

$$\underset{CS}{\le} \|\underline{u}\|^2 + \|\underline{u}\| \cdot \|\underline{v}\| + \|\underline{v}\| \cdot \|\underline{u}\| + \|\underline{v}\|^2$$

$$= (\|\underline{u}\| + \|\underline{v}\|)^2.$$

"triangle inequality":



any side of a triangle is shorter than sum of the lengths of the other two sides.

Say: $d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$ is a "metric" on V:

$d : V \times V \to \mathbb{R}_{\ge 0}$ s.t. $d(\underline{u}, \underline{v}) = d(\underline{v}, \underline{u})$,

$$\begin{cases} d(\underline{u},\underline{v})=0 \text{ iff } \underline{u}=\underline{v} \\ d(\underline{u},\underline{v})+d(\underline{v},\underline{w}) \geq d(\underline{u},\underline{w}) \end{cases}$$

(1^{st}: $\underline{v}-\underline{u}=(-1)(\underline{u}-\underline{v})$, 2^{nd}: $\|\underline{v}\|=0$ iff $\underline{v}=0$

3^{rd}: $\underline{w}-\underline{u}=(\underline{w}-\underline{v})+(\underline{v}-\underline{u})$)

Say that V is a real inner pdt space (if V is complex, $\mathrm{Re}\langle\cdot,\cdot\rangle$ gives a real inner pdt)

Then C-S says $-1 \leq \dfrac{\langle \underline{u},\underline{v}\rangle}{\|\underline{u}\|\cdot\|\underline{v}\|} \leq 1$

Def: The angle between $\underline{u},\underline{v}$ is the unique $\theta\in[0,\pi]$ s.t.

$$\langle \underline{u},\underline{v}\rangle = \|\underline{u}\|\cdot\|\underline{v}\|\cdot\cos\theta$$

(makes sense only if $\underline{u},\underline{v}\neq\underline{0}$)

Non-rigorous courses (eg. Math 200) try to define $\underline{u}\cdot\underline{v}$ by $\|\underline{u}\|\cdot\|\underline{v}\|\cdot\cos\theta$. (but this requires an independent notion of angle)

Def: $\cos\Theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\Theta^{2n}$, $\sin\Theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\Theta^{2n+1}$

$\frac{\pi}{2} = \min\{\Theta > 0 \mid \cos\Theta = 0\}$

check: $\sin^2\Theta + \cos^2\Theta = 1$, $\sin(\alpha+\beta) = \sim$

$\cos(\alpha+\beta) = \cdots$

$\Rightarrow \cos\pi = -1$, as is monotone on $[0,\pi]$ ..

Def: Call $\underline{u}, \underline{v}$ orthogonal if $\langle \underline{u}, \underline{v}\rangle = 0$, write $\underline{u} \perp \underline{v}$.

(if $\underline{u}, \underline{v} \neq 0$ this is same as saying the angle is $\pi/2$)

("Pythagoras")

Ex: if $\underline{u} \perp \underline{v}$ then $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$.

Pf: $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \cancel{\langle \underline{u}, \underline{v}\rangle} + \cancel{\langle \underline{v}, \underline{u}\rangle} + \|\underline{v}\|^2$

$= \|\underline{u}\|^2 + \|\underline{v}\|^2$

(aside: $\langle \underline{v}, \underline{u}\rangle = \overline{\langle \underline{u}, \underline{v}\rangle}$ so $\underline{u}\perp\underline{v}$ iff $\underline{v}\perp\underline{u}$)

Next time: If $(U, \langle\cdot,\cdot\rangle_U)$, $(V, \langle\cdot,\cdot\rangle_V)$ are inner pdt spaces over the same field, $\dim U = \dim V$, then have linear bijection $U \to V$ that preserves inner pdt