Math 223, lecture 30

Last time: Cauchy-Schwartz:

In an inner pdt space $(V, \langle \cdot, \cdot \rangle)$ set

$\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$ for each $\underline{v} \in V$ ("norm")

Then $|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \cdot \|\underline{v}\|$

Over any field, can define "quadratic space"
(assume char $F \neq 2$)
$1+1 \neq 0$

to be pair $(V, \langle \cdot, \cdot \rangle)$ st. (1) $\langle \underline{v}, \cdot \rangle$ is linear
(2) symmetric $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$
(3) If $\underline{u} \neq \underline{0}$, $\exists \underline{v}$ s.t. $\langle \underline{u}, \underline{v} \rangle \neq 0$

If field F has an involution ($\sigma : F \to F$ preserves $+, \cdot$, st. $\sigma^2 = 1$), can define Hermitian space by replacing (2) with

$$\langle \underline{v}, \underline{u} \rangle = \sigma(\langle \underline{u}, \underline{v} \rangle).$$

$\Rightarrow$ angle between $\underline{u}, \underline{v}$: $\cos\theta = \dfrac{\mathrm{Re}\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \cdot \|\underline{v}\|}$

Used this to show $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$.
(also $\|\alpha \underline{u}\| = |\alpha| \, \|\underline{u}\|$)

<u>Def</u>: $\underline{u} \perp \underline{v}$ ("orthogonal") if $\langle u, v \rangle = 0$

<u>Today</u>: Explore orthogonality.

<u>One message</u>: right-angled co-ordinate systems are better. : orthogonal bases better than general bases

($B \subset V$ is a basis, $B = \{\underline{v}_i\}_{i \in I}$, then the <u>co-ords</u> of $\underline{v} \in V$ are the a_i s.t. $\underline{v} = \sum_i a_i \underline{v}_i$.)

<u>Three ideas</u>: (1) Orthogonal & Orthonormal systems

(2) Gram-Schmidt procedure

(3) Orthogonal complements

Def: A set $B \subset V$ of vectors is an orthogonal system if the elements of B are non-zero and mutually orthogonal.

Lemma: An orthogonal system is independent.

Pf: Suppose $\sum_i a_i \underline{v}_i = \underline{0}$ where $\{\underline{v}_i\} \subset B$ are distinct.

For each j, we have

$$0 = \langle \underline{v}_j, \underline{0} \rangle = \langle \underline{v}_j, \sum_i a_i \underline{v}_i \rangle$$

$$\underset{\substack{\uparrow \\ \text{linearity of} \\ \langle \cdot, \cdot \rangle}}{=} \sum_i a_i \langle \underline{v}_j, \underline{v}_i \rangle \underset{\substack{\uparrow \\ \langle \underline{v}_j, \underline{v}_i \rangle = 0 \\ \text{if } i \neq j}}{=} a_j \langle \underline{v}_j, \underline{v}_j \rangle$$

But $\langle \underline{v}_j, \underline{v}_j \rangle > 0$ ($\underline{v}_j \in B$ is non-zero)

so $a_j = 0$

Lemma: let $B \subset V$ be orthogonal, let $\underline{v} \in \mathrm{Span}(B)$

Then coeff a_j of $\underline{v}$ wrt B are $\dfrac{\langle \underline{v}_j, \underline{v} \rangle}{\langle \underline{v}_j, \underline{v}_j \rangle}$.

Pf: Say $\underline{v} = \sum_i a_i \underline{v}_i$, repeat calculation to get:

$$\langle \underline{v}_j, \underline{v} \rangle = a_j \langle \underline{v}_j, \underline{v}_j \rangle.$$

Observation: a_j only depends on $\underline{v}_j$, not all of B.
(completely false for general bases)

Example: $\underline{v} = \begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{R}^2$, std. inner pdt.

$$B = \{e_1, e_2\} \quad \underline{v} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}$$

But if $B' = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$, now $\underline{v} = 0\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\1\end{pmatrix}$.

In general, determining the a_j involved solving systems of linear equations. Don't need that if system is orthogonal.

Observation: rescaling the v_i does not affect orthogonality:

$$\langle \alpha \underline{u}, \beta \underline{v} \rangle = (\bar{\alpha}\beta) \cdot \langle \underline{u}, \underline{v} \rangle.$$

Def: Call B an orthonormal system if it's orthogonal and $\langle \underline{u}, \underline{u} \rangle = 1$, i.e. $\|\underline{u}\| = 1$ for all $\underline{u} \in B$

__Def:__ The orthonormal system is __complete__ if the only vector orthogonal to all $\underline{v} \in B$ is $\underline{0}$.

__Remark:__ complete orthonormal systems are also called __orthonormal bases__, written o.n.b.

__Lemma:__ Suppose $\dim V = n < \infty$. Then every o.n.b. in V is a basis.

__Pf:__ Let $B = \{\underline{v}_i\}_{i=1}^{m} \subset V$ be an o.n.b.

For $\underline{v} \in V$ let $a_i = \langle \underline{v}_i, \underline{v} \rangle$ as above, consider

$$\underline{u} = \underline{v} - \sum_{i=1}^{m} a_i \underline{v}_i$$

Then for each j, $\langle \underline{v}_j, \underline{u} \rangle = \langle \underline{v}_j, \underline{v} - \sum_{i=1}^{m} a_i \underline{v}_i \rangle$

$$= \langle \underline{v}_j, \underline{v} \rangle - \sum_{i=1}^{m} a_i \underbrace{\langle \underline{v}_j, \underline{v}_i \rangle}_{\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}}$$

$$= \underbrace{a_j}_{\text{defn of } a_j} - a_j$$

$= 0$ so $\underline{u} \perp \underline{v}_j$ for all j, so $\underline{u} = 0$, i.e. $\underline{v} \in \mathrm{Span}(B)$

In general, let $B \subset V$ be an orthonormal system, let $\{\underline{v}_i\}_{i=1}^m \subset B$. Set $a_i = \langle \underline{v}_i, \underline{v} \rangle$ for $\underline{v} \in V$.

Then (HW) $\|\underline{v}\|^2 \geq \sum_i |a_i|^2$

("Bessel inequality") with equality iff $\underline{v} \in \mathrm{Span}(B)$ ("Parseval identity")

__Example:__ (∞-dim case) in the space $C(\mathbb{R}/\mathbb{Z})$ of continuous functions f s.t. $f(x+n) = f(x)$ if $n \in \mathbb{Z}$

Have orthonormal system $1 \cup \{\sqrt{2}\cos(2\pi kx), \sqrt{2}\sin(2\pi kx)\}_{k=1}^{\infty}$

it is complete. Better: $\{e^{2\pi ikx}\}_{k \in \mathbb{Z}}$

["Fourier series"]

__PS 2:__ Showed these are indep using $\frac{d}{dx}$.

__Now:__ compute integrals $\int_0^1 (\sqrt{2}\cos(2\pi kx))(\sqrt{2}\cos(2\pi lx))\,dx$

$$= \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \quad \ldots$$

also $\int \cos(\,)\cdot\sin(\,)$, $\int \sin(\,)\cdot\sin(\,)$, $\int 1\cdot\cos(\,)$, $\int 1\cdot\sin(\,)$.

Gram - Schmidt

Suppose we have a basis $\mathcal{B} = \{\underline{v}_i\}_{i=1}^{n} \subset V$
How do we make an orthonormal basis?

set $\underline{u}_1 = \dfrac{\underline{v}_1}{\|\underline{v}_1\|}$. Clearly $\text{Span}\{\underline{u}_1\} = \text{Span}\{\underline{v}_1\}$

Idea: remove from $\underline{v}_2$ the part we already understand.

set $\underline{w}_2 = \underline{v}_2 - \langle \underline{u}_1, \underline{v}_2 \rangle \cdot \underline{v}_1$, $\underline{u}_2 = \dfrac{\underline{w}_2}{\|\underline{w}_2\|}$

$$\langle \underline{u}_1, \underline{w}_2 \rangle = \langle \underline{u}_1, \underline{v}_2 \rangle - \langle \underline{u}_1, \underline{v}_2 \rangle \langle \underline{u}_1, \underline{u}_1 \rangle$$
$$= 0$$

so $\{\underline{u}_1, \underline{u}_2\}$ is an orthonormal system

in $\text{Span}\{\underline{v}_1, \underline{v}_2\}$ so $\{\underline{u}_1, \underline{u}_2\}$ is a basis there

In general, suppose we have $\{\underline{u}_1, \ldots, \underline{u}_k\}$ an orthonormal system in $\text{Span}\{\underline{v}_1, \ldots, \underline{v}_k\}$. set

$$\underline{w}_{k+1} = \underline{v}_{k+1} - \sum_{i=1}^{k} \langle \underline{u}_i, \underline{v}_{k+1} \rangle \underline{u}_i, \quad \underline{u}_{k+1} = \frac{\underline{w}_{k+1}}{\|\underline{w}_{k+1}\|}.$$

Then
If $1 \le j \le k$

$$\langle \underline{u}_j, \underline{w}_{k+1} \rangle = \langle \underline{u}_j, \underline{v}_{k+1} \rangle - \sum_{i=1}^{k} \langle \underline{u}_i, \underline{v}_{k+1} \rangle \overbrace{\langle \underline{u}_j, \underline{u}_i \rangle}^{\begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}}$$

$$= \langle \underline{u}_j, \underline{v}_{k+1} \rangle - \langle \underline{u}_j, \underline{v}_{k+1} \rangle = 0$$

so $\{\underline{u}_1, .., \underline{u}_k, \underline{u}_{k+1}\} \subset \mathrm{Span}\{\underline{u}_1, .., \underline{u}_k, \underline{v}_{k+1}\}$

$$= \mathrm{Span}\{\underline{v}_1, .., \underline{v}_k, \underline{v}_{k+1}\}$$

is an orthonormal system, hence a basis

($\underline{w}_{k+1} \neq \underline{0}$ by Bessel inequality,

$\underline{v}_{k+1} \notin \mathrm{Span}\{\underline{v}_1, .., \underline{v}_k\} = \mathrm{Span}\{\underline{u}_1, .., \underline{u}_k\}$)

Recap: Define
$$\boxed{\underline{w}_{k+1} = \underline{v}_{k+1} - \sum_{i=1}^{k} \langle \underline{u}_i, \underline{v}_{k+1} \rangle \underline{u}_i \qquad \underline{u}_{k+1} = \frac{\underline{w}_{k+1}}{\|\underline{w}_{k+1}\|}}$$

Then $\{\underline{u}_i\}_{i=1}^{k}$ is an orthonormal system in

$\mathrm{Span}\{\underline{v}_1, .., \underline{v}_k\}$ for $k=n$ get basis of V.

HW: do it!

<u>Observe:</u> each u_k depends on $v_1, \dots, v_k$ conversely.

So change-of-basis matrix is triangular.

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}.$$

<u>Cor:</u> If $A \in M_n(\mathbb{R})$ is invertible, apply this to colums of A, set

$A = K \cdot U$ where cols of K are an o.n.b. and U is upper-triangular

("Iwasawa decomposition")

On periodic fcns, natural to integrate on one period

For poly note $[a,b]$, $[-1,1]$ change vars

$$x = \frac{a+b}{2} + t\,\frac{a-b}{2}, \quad t \in [-1,1], \quad x \in [a,b]$$

Then $p(x) = \text{poly}(\ t\ )$

$$p(x) = x^2 = \left(\frac{a-b}{2}\right)^2 t^2 + \frac{a^2-b^2}{2} t + \left(\frac{a+b}{2}\right)^2$$