Math 223, lecture 31

V inner prod space, $\underline{u}, \underline{v} \in V$: $\underline{u} \perp \underline{v}$ iff $\langle \underline{u}, \underline{v} \rangle = 0$

orthonormal system: set B where if $\underline{u}, \underline{v} \in B$ then

$$\langle \underline{u}, \underline{v} \rangle = \begin{cases} 1 & \underline{u} = \underline{v} \\ 0 & \underline{u} \neq \underline{v} \end{cases}$$

Then if $\underline{w} \in \operatorname{Span} B$, have

$$\underline{w} = \sum_{\underline{u} \in B} \langle \underline{u}, \underline{w} \rangle \cdot \underline{u}$$

coeff computed from $\underline{u}, \underline{w}$ alone.

Gram-Schmidt procedure: Given basis, make orthonormal basis.

Examples: homework

Today: Orthogonal complements, projections.

Def: If $S \subset V$, $\underline{u} \in V$ write $\underline{u} \perp S$ if $\underline{u} \perp \underline{v}$ for all $\underline{v} \in S$. The orthogonal complement of S is

$$S^\perp = \{\underline{u} \in V \mid \underline{u} \perp S\}$$

Lemma: $S^\perp$ is a subspace

Pf: By def'n, $S^\perp = \bigcap_{\underline{v} \in S} \{\underline{v}\}^\perp$

And $\underline{v}^\perp = \{\underline{u} \mid \langle \underline{v}, \underline{u} \rangle = 0\} = \ker(\langle \underline{v}, \cdot \rangle)$

Prop: (Orthogonal decomposition) Say V is f.d.

Then $(S^\perp)^\perp = \operatorname{Span} S$, if W is a subspace then

$$W \cap W^\perp = \{\underline{0}\}, \quad V = W \oplus W^\perp.$$

Pf: If $\underline{w} \in W \cap W^\perp$ then $\underline{w} \perp \underline{w}$ so $\langle \underline{w}, \underline{w} \rangle = 0$ so $\underline{w} = \underline{0}$. Thus the sum $W + W^\perp$ is direct.

This set is complete: if $\underline{z} \perp W + W^\perp$ then $\underline{z} \perp W \subset W + W^\perp$ so $\underline{z} \in W^\perp$ then $\underline{z} \perp \underline{z}$

Saw in discussion of complete o.n. systems that this means $W \oplus W^\perp = V$.

Now $S \perp S^\perp$ so $S \subset (S^\perp)^\perp$
so $\operatorname{Span} S \subseteq (S^\perp)^\perp$ (subspace!)

suppose $(S^\perp)^\perp \supsetneq \operatorname{Span} S$.

Take o.n.b. of Span S $\{\underline{u}_1, , \underline{u}_r\}$.
let $\underline{v} \in (S^\perp)^\perp$ be indep of them.

By Gram-Schmidt can make non-zero vector $\underline{w}$
in span of $\{\underline{v}, \underline{u}_1, , \underline{u}_r\}$ which is $\perp$ to $\{\underline{u}_1, , \underline{u}_r\}$

so $\underline{w} \perp \operatorname{Span} S$ so $w \perp S$, so $\underline{w} \in S^\perp$.

But $\underline{w} \in \operatorname{Span}\{\underline{v}, \underline{u}_1, .., \underline{u}_r\} \subset (S^\perp)^\perp$.
so $\underline{w} \perp S^\perp$. So $\underline{w} \perp \underline{w}$, so $\underline{w} = \underline{0}$ $\Rightarrow\Leftarrow$

<u>Conclusion:</u> $(S^\perp)^\perp = \operatorname{Span} S$,

$$V = (\operatorname{Span} S) \oplus S^\perp$$

orthogonal

If $V = W \oplus W^\perp$, then the projections on $W, W^\perp$ are orthogonal:
if we write $\underline{v} \in V$ (uniquely!)
as $\underline{w} + \underline{w}^\perp$ then $\underline{w} \perp \underline{w}^\perp$,

so also $$\|\underline{v}\|^2 = \|\underline{w}\|^2 + \|\underline{w}^\perp\|^2.$$

<u>Example</u>: $V = \mathbb{R}^2$, $W = \mathbb{R}\underline{e}_1$, $W^\perp = \mathbb{R}\underline{e}_2$

$$V = W \oplus W^\perp = \operatorname{Span}\{\underline{e}_1, \underline{e}_2\}$$

Vector $\begin{pmatrix}1\\1\end{pmatrix} = \underline{e}_1 + \underline{e}_2$ is in neither space

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<u>Different pov</u>: If $W \subset V$ can take basis B of W, complete it to a basis C of V.
Then $W = \operatorname{Span} B$, $V = W \oplus \operatorname{Span}(C \setminus B)$.

Now assume B is orthonormal, same for C.
Then $\operatorname{span}(C \setminus B) = W^\perp$.

Eg. in $\mathbb{R}^3$, $W = \operatorname{Span}\{\underline{e}_1\}$, $W^\perp = \left\{\begin{pmatrix}0\\y\\z\end{pmatrix}\right\}$
$= \operatorname{Span}\{\underline{e}_2, \underline{e}_3\}$

$$= \operatorname{Span}\left\{\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}\right\}$$

<u>Last topic</u>: interaction of linear maps and inner products

We will examine: (1) the <u>adjoint map</u>.

(2) ways for a linear map to "respect" an inner product.

(3) Spectral theorem: (some) linear maps are diagonalizable

<u>Lemma:</u> (f.d. Riesz Representation theorem)

Let V be a f.d. inner product space, let $\varphi \in V^*$, i.e. $\varphi: V \to$ scalars is a linear map

Then there exists a unique vector $\underline{u}$ s.t.

$\forall \underline{v} \in V: \varphi(\underline{v}) = \langle \underline{u}, \underline{v} \rangle$

(aside: map $\varphi \to \underline{u}$, $V^* \to V$ is an $\mathbb{R}$-linear bijection, if scalars are complex it maps $i\varphi \to -i\underline{u}$)

Pf: If $\varphi = 0$ then $\varphi(\underline{v}) = \langle \underline{0}, \underline{v} \rangle$ for all $\underline{v}$, but if $\underline{u} \neq \underline{0}$ then $\langle \underline{v}, \underline{u} \rangle \neq \varphi(\underline{v})$

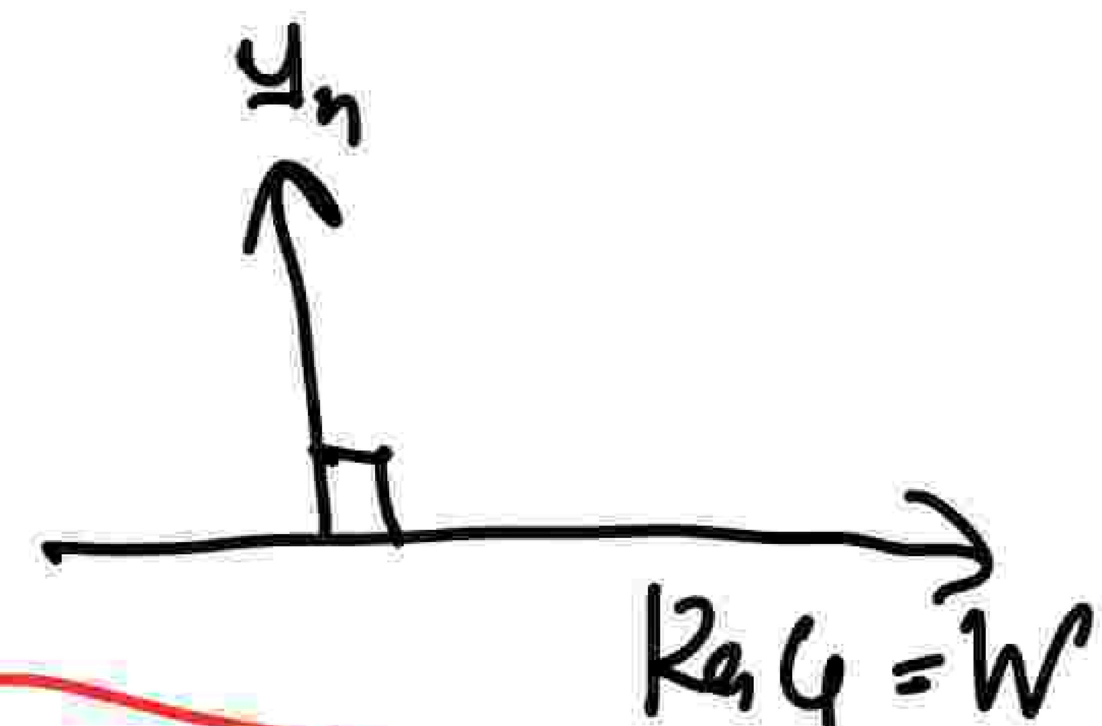
If $\varphi \neq 0$, ~~th~~ let $W = \ker \varphi$ then

$\dim W = \dim V - 1$ (rank-nullity)

Fix onb $\{\underline{u}_1, \dots, \underline{u}_{n-1}\} \subset W$, let $\underline{u}_n$ span $W^\perp$.
$n = \dim V$, $\|\underline{u}_n\| = 1$

Let $\underline{u} = \overline{\varphi(\underline{u}_n)} \cdot \underline{u}_n$. Then $\varphi(\underline{v}) = \langle \underline{u}, \underline{v} \rangle$:

say $\underline{v} = \sum_{i=1}^{n} a_i \underline{v}_i$.



Then $\varphi(\underline{v}) = \sum_{i=1}^{n} a_i \varphi(\underline{v}_i) = a_n \varphi(\underline{u}_n)$

$$\langle \underline{u}, \underline{v} \rangle = \langle \overline{\varphi(\underline{u}_n)}\, \underline{u}_n, \sum_{i=1}^{n} a_i \underline{v}_i \rangle$$

$$= \varphi(\underline{u}_n) \cdot \sum_{i=1}^{n} a_i \langle \underline{u}_n, \underline{u}_i \rangle$$

$$= a_i \varphi(\underline{u}_n)$$

Uniqueness: If $\langle \underline{u}, \underline{v} \rangle = \langle \underline{u}', \underline{v} \rangle$ for all $\underline{v}$, then $\langle \underline{u}_0 - \underline{u}', \underline{v} \rangle = 0$ for all $\underline{v}$.

so $\langle \underline{u} - \underline{u}', \underline{u} - \underline{u}' \rangle = 0$ so $\underline{u} - \underline{u}' = \underline{0}$ so $\underline{u} = \underline{u}'$.

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(anti) linearity: HW.

<u>Example:</u> $V = \mathbb{R}[x]^{<n}$, inner pdt $\int_{-1}^{1} \bar{f} g \, dx$

Map $\varphi(f) = f(0)$ is linear.

<u>Lemma:</u> $\exists$ polynomial g of $\deg < n$ s.t. if $\deg f < n$ then

$$\int_{-1}^{1} \overline{g(x)} f(x) \, dx = f(0)$$

<u>Examples:</u> 1) If A is a matrix, can try to define inner pdt by

$$\langle \underline{u}, \underline{v} \rangle_A = \underline{u}^T A \underline{v}$$

works if A is "positive definite" symmetric)

2) $\langle \underline{u}, A\underline{v} \rangle = \underline{u}^T \cdot (A\underline{v}) = (\underline{u}^T A) \cdot \underline{v}$

std inner pdt on $\mathbb{R}^n$

$$= (A^T \underline{v})^T \cdot \underline{v} = \langle A^T \underline{u}, \underline{v} \rangle$$

$(XY)^T = Y^T X^T$

$(A^T)^T = A$