Math 223, lecture 32

Last time: (1) Orthogonal complements:

$$W^{\perp} = \{ \underline{v} \in V \mid \exists \underline{w} \in W : \underline{v} \perp \underline{w} \}$$

$$= \bigcap_{\underline{w} \in W} \underline{w}^{\perp} = \bigcap_{\underline{w} \in W} \mathrm{Ker}(\langle \underline{w}, \cdot \rangle)$$

linear functional!

(call $\varphi_{\underline{w}}$)

checked: $W \cap W^{\perp} = \{\underline{0}\}$

and (V f.d.) $W \oplus W^{\perp} = V$.

(every $\underline{v} \in V$ has a unique representation

$$\underline{v} = \underline{w} + \underline{u}, \quad \underline{w} \in W, \ \underline{u} \in W^{\perp})$$

Later: W = eigenspaces of an appropriate linear map

(2) Riesz Representation thm: every functional on V is of the form $\varphi_{\underline{w}}$ for some $\underline{w} \in V$.
(V f.d.)

Proved this last time HW: maps $\underline{w} \to \varphi_{\underline{w}}$, $\varphi_{\underline{w}} \to \underline{w}$ are additive, anti-linear: $c\underline{w} \to \varphi_{c\underline{w}} = \bar{c} \cdot \varphi_{\underline{w}}$.

<u>Today</u>: talk about the <u>adjoint</u> of a linear map.

<u>Example</u>: Say $V = \mathbb{R}^n$, std inner product:

$$\langle \underline{u}, \underline{v} \rangle = \underline{u}^T \cdot \underline{v}$$

think of u as an $n \times 1$ matrix, take transpose

matrix mult. $M_{1,n}(\mathbb{R}) \times M_{n,1}(\mathbb{R}) \to M_{1,1}(\mathbb{R})$

Thus: if $A \in M_n(\mathbb{R})$,

$$\langle \underline{u}, A\underline{v} \rangle = \underline{u}^T \cdot A\underline{v} = (\underbrace{\underline{u}^T}_{1\times n \text{ matrix}} \cdot \underbrace{A}_{n\times n \text{ matrix}}) \cdot \underbrace{\underline{v}}_{n \times 1 \text{ matrix}}$$

$$\underset{A = (A^T)^T}{=} [\underline{u}^T (A^T)^T] \cdot \underline{v} \underset{(AB)^T = B^T A^T}{=} (A^T \underline{u})^T \cdot \underline{v} = \langle A^T \underline{u}, \underline{v} \rangle$$

i.e. A^T is the linear map s.t.

$$\langle \underline{u}, A\underline{v} \rangle = \langle A^T \underline{u}, \underline{v} \rangle$$

In general, let $(V, \langle \cdot, \cdot \rangle)$ be a f.d. inner prdt space. Let $T \in End(V)$.

<u>Def</u>: The <u>adjoint</u> map $T^* = T^\dagger \in End(V)$ is the map s.t.

$$\langle \underline{u}, T\underline{v} \rangle = \langle T^*\underline{u}, \underline{v} \rangle$$

for all $u, \underline{v} \in V$.

<u>Prop</u>: The adjoint exists, is unique.

<u>Pf</u>: Observe that the map $\underline{v} \mapsto \langle u, T\underline{v} \rangle$ is linear in $\underline{v}$: this is $\varphi_{\underline{u}} \circ T$.

By the Riesz Representation thm, have a <u>unique</u> vector $T^*\underline{u}$ s.t.

$$\varphi_{T^*\underline{u}} = \varphi_{\underline{u}} \circ T$$

This shows that there is a unique <u>function</u> $T^*: V \to V$ s.t. $\langle \underline{u}, T\underline{v} \rangle = \langle T^*\underline{u}, \underline{v} \rangle$ for all $u, \underline{v}$.

Want also: T^* is linear. Indeed:

$$\langle T^*(\alpha \underline{u} + u'), \underline{v} \rangle \underset{\text{def of } T^*}{=} \langle \alpha \underline{u} + \underline{u}', T\underline{v} \rangle$$

$$\underset{\substack{\uparrow \\ \text{linearity of } \langle \cdot,\cdot \rangle}}{=} \bar{\alpha} \langle \underline{u}, T\underline{v} \rangle + \langle \underline{u}', T\underline{v} \rangle = \bar{\alpha} \langle T^* \underline{u}, \underline{v} \rangle + \langle T^* \underline{u}', \underline{v} \rangle$$

$$\overset{b}{=} \langle \alpha T^* \underline{u} + T^* \underline{u}', \underline{v} \rangle$$

So $T^*(\alpha \underline{u} + \underline{u}')$, $\alpha T^* \underline{u} + T^* \underline{u}'$ have same inner pdt with all $\underline{v}$, so they are equal ■

Example: $V = \mathbb{C}^n$, std Hermitian product:

$$\langle \underline{u}, \underline{v} \rangle = \underline{u}^\dagger \cdot \underline{v} = \sum_i \bar{u}_i v_i =$$

$$A^\dagger_{ij} = \bar{A}_{ji}.$$

Same computation as before: adjoint of a matrix $A \in M_n(\mathbb{C})$, is $A^\dagger = \bar{A}^T$.

Example: Interpret the formula for integration by parts as saying if $D = \frac{d}{dx}$ then $D^* = -D$:

$$\int \bar{f}\, Dg = -\int \overline{(Df)} \cdot g$$

(true only with boundary conditions dealing with boundary terms)

Also: (Ex): $(S+T)^* = S^* + T^*$
$(cT)^* = \bar{c}T^*$

$\Rightarrow$ Physicists define $\hat{p} = -i\hbar \frac{d}{dx}$ $\hbar$ = constant

$(\hat{p})^* = i\hbar\left(-\frac{d}{dx}\right) = \hat{p}$.

"momentum operator".

($\hat{p}$ is "self-adjoint").

Define $\hat{x} = M_x$ $\quad ((\hat{x}\cdot f)(x) = f(x)\,x)$

also s.a. : check $(M_f)^* = M_{\bar{f}}$. Also s.a

HW: Classical mechanics, energy of a spring system ("harmonic oscillator) is:

$$E = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2$$

kinetic $\quad$ potential energies

$\Rightarrow$ QM energy operator for quantum harmonic oscillator is

$$\frac{1}{2m}\hat{p}^2 + \frac{1}{2}k\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}k M_{x^2}$$

HW: spectrum of this operator.

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^* = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix}^* = \overline{\begin{pmatrix} 1 & -2i \\ 2i & 4 \end{pmatrix}} = \begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix}$$

this is a self-adjoint map

Def: Call a matrix $A \in M_n(\mathbb{C})$ symmetric if $A^T = A$, Hermitian if $\bar{A}^T = A$ i.e. $A^T = \bar{A}$.

Real symmetric / complex hermitian matrices are self-adjoint maps wrt std inner pdt.

Big thm:

Thm: (f.d. spectral thm for self-adjoint maps)

Let $T \in End(V)$ be a self-adjoint map on a f.d inner pdt spaces

Then V has an o.n.b. consisting of eigenvectors of T. In particular, T is diagonalizable

<u>Example:</u> $A = \begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix}$ $\quad \operatorname{tr} A = 5$

$\det A = 4 + (2i)^2 = 4(1-1) = 0$

so char poly is

$$p_A(t) = t^2 - 5t = t(t-5)$$

(If ev. are λ, μ then $p_A(t) = (t-\lambda)(t-\mu)$

$$= t^2 - \underset{\operatorname{tr} A}{\underset{\uparrow}{(\lambda+\mu)}}t + \underset{\det A}{\underset{\uparrow}{\lambda\mu}}$$)

so eigenvalues are $0, 5$

(next time: eigenvalues must be real)

<u>eigenvectors:</u> $\begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ eg. $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$

$\begin{pmatrix} -4 & 2i \\ -2i & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ eg. $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$

<u>check:</u>

$$\begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix}\begin{pmatrix} -2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} 1-(2i)^2 \\ -2i-8i \end{pmatrix} = \begin{pmatrix} 5 \\ 10i \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\left\langle \begin{pmatrix} -2i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2i \end{pmatrix} \right\rangle = (2i, 1) \cdot \begin{pmatrix} 1 \\ -2i \end{pmatrix} = 2i - 2i = 0$$

o.n.b. of eigenvectors is $\left\{ \begin{pmatrix} -\frac{2i}{\sqrt{5}} \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{5} \\ -2i/\sqrt{5} \end{pmatrix} \right\}$.

$\left\| \begin{pmatrix} 1 \\ -2i \end{pmatrix} \right\| = \sqrt{|1|^2 + |2i|^2} = \sqrt{1+4} = \sqrt{5}$

if $S = \begin{pmatrix} -\frac{2i}{\sqrt{5}} & 1/\sqrt{5} \\ 1/\sqrt{5} & -\frac{2i}{\sqrt{5}} \end{pmatrix}$ then $S^* = S^{-1}$.

"unitary" matrix

so $A = S \begin{pmatrix} 0 & \\ & 5 \end{pmatrix} S^*$, here $S^\dagger = \begin{pmatrix} \frac{2i}{\sqrt{5}} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2i/\sqrt{5} \end{pmatrix}$

<u>Ex</u> (Plancherel identity):

V inner prdt space, $\{u_i\}_{i=1}^n \subset V$ o.n.b.

$\underline{v} = \sum_{i=1}^n a_i u_i$, $\underline{w} = \sum_{j=1}^n b_j u_j$

Then $\langle \underline{v}, \underline{w} \rangle = \sum_{i=1}^n \overline{a_i}\, b_i$

(wrt basis $\{u_i\}$, if matrix of T is A
o.n.b.
matrix of T^* is $A^\dagger$.)

<u>aside</u>

If V, W general vsp, $T \in \mathrm{Hom}(V, W)$

define $T^*: W^* \to V^*$ by $T^*\varphi = \varphi \circ T$

(with inner pdt, can identify $V \leftrightarrow V^*$, $W \leftrightarrow W^*$
then T^* becomes map $W \to V$)

If $\{\underline{v}_i\} \subset V$, $\{\underline{w}_j\} \subset W$ bases

A matrix of T wrt them

Then A^T is matrix of T^* wrt <u>dual bases</u>

$\{\underline{v}_i^*\} \subset V^*$, $\{w_j^*\} \subset W^*$,

(specific subsets of V^*, W^*)

$$(ST)^* \cdot \varphi = \varphi \circ (ST) = (\varphi \circ S) \circ T = T^*(\varphi \circ S)$$
$$= T^* S^* \varphi$$

so $(ST)^* = T^* S^*$