Math 223, lecture 33

Last time: Adjoint: $\langle \underline{u}, T\underline{v} \rangle = \langle T^* \underline{u}, \underline{v} \rangle$

A is self-adjoint if $T^* = T$

If $B \subset V$ is an orthonormal basis, matrix of T wrt B is A, then matrix of T^* wrt B is $A^\dagger$ where

$$A^\dagger_{ij} = \overline{A_{ji}}.$$

In general, T^* depends on $\langle \cdot, \cdot \rangle$.

recall that A_{ij} obtained by decomposing $T\underline{u}_j$ wrt basis $B = \{\underline{u}_i\}_{i=1}^n \subset V$.

so (if B is orthonormal) $A_{ij} = \langle \underline{u}_i, T\underline{u}_j \rangle$

then $A_{ij} = \langle T^* \underline{u}_i, \underline{u}_j \rangle = \overline{\langle \underline{u}_j, T^* \underline{u}_i \rangle}$.

Today: Prove spectral thm for selfadjoint maps on f.d. inner prdt spaces

1. Eigenvalues & eigenvectors
2. Proof of thm.

Fix inner pdt space $(V, \langle\cdot,\cdot\rangle)$, self adjoint map $T \in \operatorname{End}(V)$

<u>Lemma</u>: Suppose $T\underline{v} = \lambda \underline{v}$, $\underline{v} \neq \underline{0}$. Then λ is real.

<u>Pf</u>: Evaluate $\langle \underline{v}, T\underline{v} \rangle$ in two ways

(1) $\langle \underline{v}, T\underline{v} \rangle = \langle \underline{v}, \lambda \underline{v} \rangle = \lambda \langle \underline{v}, \underline{v} \rangle$.

(2) $\langle \underline{v}, T\underline{v} \rangle \underset{\text{def of } T^*}{=} \langle T^*\underline{v}, \underline{v} \rangle \underset{\text{hyp } T^* = T}{=} \langle T\underline{v}, \underline{v} \rangle =$

$\underset{\text{inner pdt}}{=} \overline{\langle \underline{v}, T\underline{v} \rangle} = \overline{\lambda \langle \underline{v}, \underline{v} \rangle} \underset{\langle \underline{v}, \underline{v} \rangle \in \mathbb{R}}{=} \bar{\lambda} \langle \underline{v}, \underline{v} \rangle$

So $\lambda \langle \underline{v}, \underline{v} \rangle = \bar{\lambda} \langle \underline{v}, \underline{v} \rangle$

Now $\underline{v} \neq \underline{0} \Rightarrow \langle \underline{v}, \underline{v} \rangle \neq 0 \Rightarrow \lambda = \bar{\lambda}$. ∎

<u>Example</u>: If $A \in M_n(\mathbb{R})$ is <u>symmetric</u> $A = A^T$ then all eigenvalues of A are real.

<u>Lemma:</u> Let $\lambda \neq \mu$ then eigenspaces V_λ, V_μ are orthogonal.

<u>Pf:</u> Say $T\underline{v} = \lambda \underline{v}$, $T\underline{w} = \mu \underline{w}$

Now examine $\langle \underline{v}, T\underline{w} \rangle = \langle \underline{v}, \mu \underline{w} \rangle = \mu \langle \underline{v}, \underline{w} \rangle$

$= \langle T^* \underline{v}, \underline{w} \rangle = \langle T\underline{v}, \underline{w} \rangle = \langle \lambda \underline{v}, \underline{w} \rangle = \lambda \langle \underline{v}, \underline{w} \rangle$

λ is real

so $(\lambda - \mu) \cdot \langle \underline{v}, \underline{w} \rangle = 0$

if $\lambda \neq \mu$, $\langle \underline{v}, \underline{w} \rangle = 0$. ∎

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② Spectral thm

<u>Lemma:</u> T has at least one eigenvalue with eigenvector in V.

<u>Pf 1:</u> We saw that any $T \in \mathrm{End}(V)$ has a complex eigenvalue, but all eigenvalues are real, so have a real eigenvalue λ. Then $T - \lambda$ not invertible, so has kernel in V.

(system $(T-\lambda)\underline{v} = \underline{0}$ has complex solutions, if coeffs are all real then has real solutions too)

Pf 2: Consider the function $f : V \to \mathbb{R}$ given by $f(\underline{v}) = \langle \underline{v}, T\underline{v} \rangle$

$$\left(\langle \underline{v}, T\underline{v} \rangle \underset{T^* = T}{=} \langle T\underline{v}, \underline{v} \rangle \underset{\text{inner prod}}{=} \overline{\langle \underline{v}, T\underline{v} \rangle} \right)$$

(if $\underline{v} = \sum_i x_i \underline{u}_i$, $\{\underline{u}_i\}$ = basis, this is a poly in the x_i)

Restrict to the sphere $\{\underline{v} \mid \|\underline{v}\| = 1\}$

Calculus: it has a maximum, if max at $\underline{v}$ then theory of Lagrange multipliers give

$$(\nabla f)(\underline{v}) = \lambda \cdot \nabla(\|\underline{v}\|^2 - 1)$$

$$\Updownarrow$$

$$2T\underline{v} = 2\lambda\underline{v}.$$

$\blacksquare$

(details in notes)

Def: Call a subspace $W \subset V$ <u>T-invariant</u> if $TW \subset W$.

(so $T\restriction_W \in End(W)$)

<u>Lemma:</u> Let W be T-invariant. Then so is $W^\perp$.

<u>Pf:</u> Let $\underline{v} \in W^\perp$, to check if $T\underline{v} \in W^\perp$ let $\underline{w} \in W$ then

$$\langle \underline{w}, T\underline{v} \rangle \underset{T^* = T}{=} \langle T\underline{w}, \underline{v} \rangle \underset{\substack{T\underline{w} \in W,\ \underline{v} \in W^\perp \\ \text{so } T\underline{w} \perp \underline{v}}}{=} 0$$

$\blacksquare$

<u>Theorem:</u> Let V be a f.d. inner product space. Let $T \in End(V)$ be self adjoint. Then there is a o.n.b. of V consisting of eigenvectors of T

<u>Pf:</u> Let $B \subset V$ be a maximal orthonormal system consisting of eigenvectors of T.

(eg. empty set not maximal if $\dim V > 0$ since T has an eigenvector)

Suppose B is incomplete. Let $W = \operatorname{Span} B$, then $W^\perp \neq \{\underline{0}\}$. W is T-invariant: $T(B) \subset W$. So $TW \subset W$. (if $T\underline{u}_i = \lambda_i \underline{u}_i$ then $T(\sum_i \alpha_i \underline{u}_i) = \sum_i (\alpha_i \lambda_i) \underline{u}_i$)

By Lemma 2, $T(W^\perp) \subset W^\perp$.

Then $T\restriction_{W^\perp} \in \mathrm{End}(W^\perp)$. It's still self-adjoint:

$\langle \underline{u}, T\underline{v} \rangle = \langle T\underline{u}, \underline{v} \rangle$ holds in V, hence in $W^\perp$.

By Lemma 1, $T\restriction_{W^\perp}$ has an eigenvector, $\underline{v}$, which we can choose normalized.

But then $\underline{v} \perp B$, so $B \cup \{\underline{v}\}$ is a larger o.n. system consisting of eigenvectors

$\Rightarrow\Leftarrow$

■

<u>Ex:</u> A real 2×2 matrix can be orthogonally diagonalized

$A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ $\quad \mathrm{tr}\, A = 3, \ \det A = -7$

$$p_A(t) = t^2 - 3t - 7 \ , \quad \lambda_{1,2} = \frac{3 \pm \sqrt{37}}{2}$$

.. find eigenvectors, .

if $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ $\quad p_A(t) = t^2 - (a+d)t + (ad - b^2)$

$$\Delta = (a+d)^2 - 4(ad - b^2) = a^2 - 2ad + d^2 + b^2$$
$$= (a-d)^2 + b^2 \geq 0$$

One more Idea: Orthogonal / unitary maps

$\{u_i\}$ basis of V (o.n.b.), A matrix of T w.r.t this basis

$\{v_i\}$ o.n.b. of eigenvectors, $D =$ " " " " $\{v_i\}$

$A = SDS^{-1}$, j'th columns of S = expansion of v_j in $\{u_i\}$

$$S_{ij} = \langle u_i, v_j \rangle$$

conversely, S^{-1} = expansion of $\{u_i\}$ in $\{v_i\}$

$$S^{-1}_{ij} = \langle v_j, u_i \rangle$$

$$\Rightarrow \quad S^{-1} = S^{\dagger}.$$

Easy to compute S^{-1}: rows are complex conjugates of the eigenvectors.

__Def:__ $S \in M_n(\mathbb{R})$ s.t. $S^{-1} = S^{T}$ ($SS^{T} = I_n$) is called __orthogonal__

$S \in M_n(\mathbb{C})$ s.t. $S^{-1} = S^{\dagger}$ ($SS^{\dagger} = I_n$) is called __unitary__.

(if $T^* = T^{-1}$ then:

$\langle T\underline{u}, T\underline{v} \rangle = \langle T^* T\underline{u}, \underline{v} \rangle = \langle \underline{u}, \underline{v} \rangle$

so T preserves inner pdts)

(Eg. in QM, quantum state is a vector in V, "observables" = selfadjoint operators, time evolution = unitary map)]

$A \in M_{n,m}(F)$, nullspace $= \dim_F \{ \underline{x} \in F^m \mid A\underline{x} = \underline{0} \}$

extend scalars to $K \supseteq F$. Then $\dim_K \{ \underline{x} \in K^m \mid A\underline{x} = \underline{0} \}$ is same.

Pf: this is determined by # pivots in row echelon form