Math 223, lecture 34

Last time: Spectral Theorem

If $A \in M_n(\mathbb{R})$ is symmetric, or $A \in M_n(\mathbb{C})$ is hermitian ($A^T = \bar{A}$), more generally, if $T \in \mathrm{End}(V)$ is self-adjoint (V = f.d. inner product space)

Then $\mathbb{R}^n / \mathbb{C}^n / V$ has an orthonormal basis consisting of eigenvectors of A / T.

$\Rightarrow$ $A = SDS^{-1} = SDS^{\dagger}$ where S is an orthogonal / unitary map, D = diagonal

Pf: find one eigenvector, restrict to orthogonal complement, recurse.

If $(V, \langle\cdot,\cdot\rangle)$ is an inner product space, a map $U \in \mathrm{End}(V)$ is orthogonal / unitary ($\mathbb{R}$, $\mathbb{C}$ scalars resp.)

$$\text{if } \langle U\underline{u}, U\underline{v}\rangle = \langle \underline{u}, \underline{v}\rangle \text{ for all } \underline{u}, \underline{v}$$

$$\Leftrightarrow U^* = U^{-1} \Leftrightarrow \|U\underline{u}\| = \|\underline{u}\|$$

Polarization identity: (real variable version)

$$\|\underline{u} \pm \underline{v}\|^2 = \langle \underline{u} \pm \underline{v}, \underline{u} \pm \underline{v} \rangle = \|\underline{u}\|^2 \pm 2\langle \underline{u}, \underline{v}\rangle + \|\underline{v}\|^2$$

$$\Downarrow$$

$$4\langle \underline{u}, \underline{v} \rangle = \|\underline{u} + \underline{v}\|^2 - \|\underline{u} - \underline{v}\|^2$$

matrix is unitary if $U^{+} = U^{-1}$, $\Leftrightarrow$

columns are an onb

$\Leftrightarrow$ rows " " ".

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commutetivity & the spectral theorem

if A^{k} is diagonalble, is A diagonalble?

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iff $A = SDS^{-1}$

colums of S are eigenvectors so:

$S\underline{e}_i$ = i'th eigenvector $= \underline{v}_i$

S^{-1}(i'th eignen vector) $= \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix} \leftarrow i$

$$(SDS^{-1})(\underline{v}_i) = SD\underline{e}_i \underset{\substack{\uparrow \\ D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}}{=} S\lambda_i \underline{e}_i = \lambda_i S\underline{e}_i = \lambda_i \underline{v}_i$$

$$S^{-1}(\underline{u}) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ s.t. } \underline{u} = \sum_{i=1}^{n} a_i \underline{v}_i$$

$$S\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} a_i \underline{v}_i$$

Exercises with adjoints, spectral theorem

Observe: $(ST)^* = T^* S^*$!

$$\langle \underline{u}, ST\underline{v} \rangle = \langle S^*\underline{u}, T\underline{v} \rangle = \langle T^* S^* \underline{u}, \underline{v} \rangle$$

$$\parallel$$

$$\langle (ST)^* \underline{u}, \underline{v} \rangle$$

so if S, T are self adjoint, $(ST)^* = TS$ which need not be ST.

So ST is self adjoint iff $ST = TS$

if S, T diagonal wrt same basis (have common basis of eigenvectors) then $TS = ST$

(diagonal matrices commute)

Conversely, if $ST = TS$ we saw: S acts on each eigenspace of T:

if $T\underline{v} = \lambda v$ then $T(S\underline{v}) = \lambda(S\underline{v})$

so if S, T selfadjoint, spectral theorem ~~for~~ for T says:

$$V = \bigoplus_{\substack{\lambda \text{ e.v.} \\ \text{of } T}} V_\lambda$$

spectral theorem for S in <u>each</u> V_λ

says: Can choose basis for each V_λ consisting of eigenvectors of S.

These are then joint eigenvectors, so S, T can be diagonalized simultaneously.

<u>Observe:</u> if U unitary: $U^* = U^{-1}$, D = diagonal real,

$$(UDU^{-1})^* = (UDU^*)^* = (U^*)^* D^* U^* \underset{\substack{\uparrow \\ D = D^*}}{=} UDU^*$$

so UDU^{-1} is selfadjoint.

If D is diagonal, not real, then

$(UDU^{-1})^* = UD^*U^{-1}$

so commutes with UDU^{-1}.

converse to this: Def: Call T normal if T, T^* commute

Spectral thm: T is normal iff V has an onb of eigenvalues of T.

(saw: If onb exists $T = UDU^{-1}$, U = unitary, D = diag $\Rightarrow T^*$ commutes with T)

other direction: define $T_+ = \frac{T+T^*}{2}$

$T_- = \frac{T-T^*}{2i}$

then T_+, T_- are selfadjoint.

$(T_+)^* = \frac{T^*+T}{2} = T_+$, $(T_-^*) = \frac{T^*-T}{2(-i)} = T_-$

T_+, T_- commute iff T, T^* do.

If T, T^* commute, T_+, T_- commute

$\Rightarrow$ diagonalize together, then $T = T_+ + iT_-$

so this diagonalises T. ∎

Ex: U orthogonal / unitary

$U^* = U^{-1}$ so $UU^* = Id = U^*U$

$\Rightarrow U$ is unitarily diagonable

Ex: $\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} = R(\theta)$ orthogonal:

$\left\langle \begin{pmatrix}\cos\theta \\ -\sin\theta\end{pmatrix}, \begin{pmatrix}\sin\theta \\ \cos\theta\end{pmatrix} \right\rangle = 0, \quad \cos^2\theta + \sin^2\theta = 1$

eigenvalues are $\cos\theta \pm i\sin\theta$

$R(\theta) = \begin{pmatrix}\cos\theta & \\ & \cos\theta\end{pmatrix} + i\begin{pmatrix} & \frac{\sin\theta}{i} \\ -\frac{\sin\theta}{i} & \end{pmatrix}$

$= \cos\theta \begin{pmatrix}1 & \\ & 1\end{pmatrix} + \sin\theta \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

Hom(U,V) vsp

Hom(V,U): check that

$\langle A, B\rangle = \mathrm{tr}(A^*B)$ is an inner prod on End(V)