Lior Silberman's Math 501: Problem Set 4 (due 9/10/2020)

Prime fields and the characteristic

1. Let R be a ring.
   (a) Show that there is a unique ring homomorphism $\varphi\colon \mathbb{Z} \to R$. We generally identify $n \in \mathbb{Z}$ with $\varphi(n) \in R$.
   (b) Let $p \geq 0$ be such that $\mathrm{Ker}(\varphi) = (p)$. If R is a field show that either $p = 0$ or p is prime.

   DEFINITION. We call p the *characteristic* of the field.

   (c) Let K be a field of characteristic $p > 0$. Show that the image of φ is the minimal subfield of K ("prime subfield"), and that it is isomorphic to the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
   (*c) Let K be a finite field. Show that there exists a prime p and a natural number n so that $|K| = p^n$.
   (d) Let K be a field of characteristic zero. Show that there is a unique homomorphism $\mathbb{Q} \hookrightarrow K$ and conclude that the minimal subfield ("prime subfield") of K is isomorphic to $\mathbb{Q}$.

Quadratic fields

Let K be a field of characteristic not equal to 2. Write $K^\times$ for the multiplicative group of K, $(K^\times)^2$ for its subgroup of squares.

2. (Reduction to squares) Let $L : K$ be an extension of degree 2.
   (a) Show that there exists $\alpha \in L$ such that $K(\alpha) = L$. What is the degree of the minimal polynomial of α?
   (b) Show that we can choose α so that $\alpha^2 = d \in K^\times$, in which case $L : K$ is isomorphic to $K(\sqrt{d}) : K$.

3. (Classifying the extensions)
   (a) Assume that $d \in K^\times$ is not a square. Show that $e \in K$ is a square in $K(\sqrt{d})$ iff $e = df^2$ for some $f \in K$. Where did you use the assumption about the characteristic?
   (b) Show that the extensions $K(\sqrt{d})$ and $K(\sqrt{e})$ are isomorphic iff $\frac{d}{e} \in (K^\times)^2$ (in general, the isomorphism will not send $\sqrt{d}$ to $\sqrt{e}$).
   Hint: Construct a K-homomorphism $K(\sqrt{e}) \to K(\sqrt{d})$. Why is it surjective? Injective?
   (c) Show that quadratic extensions of K are in bijection with non-trivial elements of the group $K^\times/(K^\times)^2$.
   RMK Note that $\mathbb{R}^\times/(\mathbb{R}^\times)^2 \simeq \{\pm 1\}$, so the real numbers have a unique quadratic extension. See also the first supplementary problems to PS3.