4. Continuity; The Intermediate Value Theorem (21/9/2021)

Goals.

(1) Continuity
  (a) Definition
  (b) Gluing of functions

(2) The Intermediate Value Theorem
  (a) With given endpoints
  (b) Free-form

(3) ~~The derivative~~
  (a) ~~Definition~~
  (b) ~~Some calculations~~

Last Time. limits at ∞, infinite limits

Ex: $\lim_{x\to 0} \frac{\sin x - 5}{x^2(x+2)} = \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\sin x - 5}{x+2}\right)$

$x^2 \xrightarrow[x\to 0]{} 0$, while $\frac{\sin x - 5}{x+2} \xrightarrow[x\to 0]{} \frac{0-5}{0+2} = -5/2 < 0$

So expression blows up. Since $\frac{\sin x - 5}{x+2} < 0$ near 0, $\frac{1}{x^2} > 0$, $\lim_{x\to 0} \frac{\sin x - 5}{x^2(x+2)} = -\infty$.

Example:

① $\dfrac{x^3+5}{x^2+2x} = \dfrac{x^3}{x^2} \cdot \dfrac{(1+5/x^3)}{(1+2/x)} = x \cdot \dfrac{1+5/x^3}{1+2/x} \xrightarrow[x\to\infty]{} \infty$

② $\dfrac{x^2+2x}{3x^3+5} = \dfrac{x^2}{x^3} \cdot \dfrac{1+2/x}{3+5/x^3} = \dfrac{1}{x} \cdot \dfrac{1+2/x}{3+5/x^3} \xrightarrow[x\to-\infty]{} 0 \cdot \dfrac{1+0}{3+0} = 0$

③ $\dfrac{3x^4+5x^2}{7x^4+3x} = \dfrac{x^4}{x^4} \cdot \dfrac{3+5/x^2}{7+3/x^3} = \dfrac{3+5/x^2}{7+3/x^3} \xrightarrow[x\to\infty]{} \dfrac{3+0}{7+0}$

"thought"/scrap paper:

$\dfrac{3x^4+5x^3}{7x^4+3x} \sim_\infty \dfrac{3x^4}{7x^4} = \dfrac{3}{7}$

③ $\lim\limits_{x\to\infty} \dfrac{3x^4+5x^2}{7x^4+3x} = \lim\limits_{x\to\infty} \dfrac{3+5/x^2}{7+3/x^3} = \dfrac{3+0}{7+0} = \dfrac{3}{7}$

though:

$\dfrac{\sin x - 5}{x^2(x+2)} \sim_0 \dfrac{0-5}{x^2(0+2)} = -5/2 \ \dfrac{1}{x^2}$

(aside: $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2} = ?$)

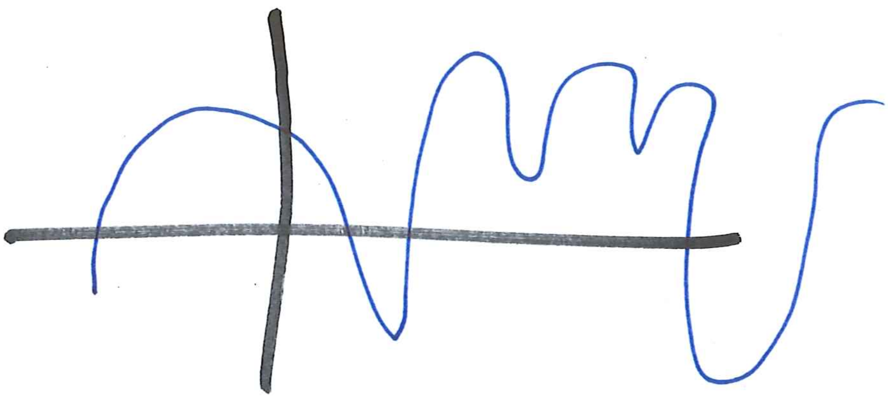
Continuity

Idea: "no breaks in graph".

Def: f is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

$$\Longleftrightarrow \lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(a)$$



cts

discontinuous

Promise: Any formula is cts where defined

Math 100 – WORKSHEET 4
CONTINUITY; THE IVT

1. Continuity

(1) Which of these functions are continuous everywhere? Why?

(a) $f(x) = \begin{cases} x & x < 0 \\ \cos x & x \geq 0 \end{cases}$

On $(-\infty,0)$ and on $(0,\infty)$ f is defined by formula, hence cts

At $x=0$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$ $\neq$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = \cos 0 = 1 = f(0)$

so f is discontinuous at $x=0$

(b) $g(x) = \begin{cases} x & x < 0 \\ \sin x & x \geq 0 \end{cases}$

On $(-\infty,0)$, $(0,\infty)$ g is defined by formula, hence cts.

At $x=0$ $\lim_{x\to 0^-} g(x) = \lim_{x\to 0} x = 0$ $=$ $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \sin x = \sin 0 = 0 = g(0)$

So g is cts everywhere.

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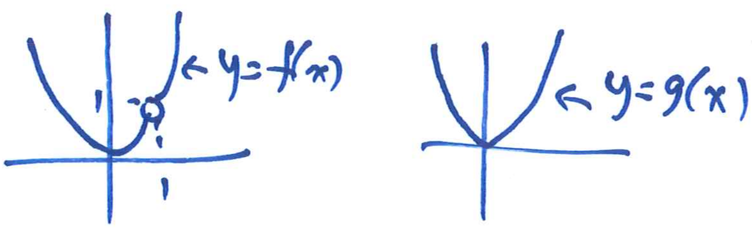
(2) Let $f(x) = \frac{x^3 - x^2}{x-1}$.

(a) Why is $f(x)$ discontinuous at $x = 1$? because its undefined there.

(b) Find b such that $g(x) = \begin{cases} f(x) & x \neq 1 \\ b & x = 1 \end{cases}$ is continuous everywhere.

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^3-x^2}{x-1} = \lim_{x\to 1} \frac{x^2(x-1)}{x-1} = \lim_{x\to 1} x^2 = 1$$

So g is cts if $b=1$



(c) Find c, d such that $h(x) = \begin{cases} \sqrt{x} & 0 \leq x < 1 \\ c & x = 1 \\ d - x^2 & x > 1 \end{cases}$

is continuous.

On $[0,1)$, $(1,\infty)$ h is defined by formula, hence cts

At 1: $\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$ $\qquad h(1) = c$

$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} (d - x^2) = d - 1^2 = d - 1$

So h is cts at 1 if $1 = c = d-1$, i.e. if $\boxed{c=1, d=2}$

id est = that is

'story': we'll take $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, equate them, get equation for c, solve the equation.

(d) (Final 2013) For which value of the constant c is $f(x) = \begin{cases} cx^2+3 & x \geq 1 \\ 2x^3 - c & x < 1 \end{cases}$ continuous on $(-\infty, \infty)$?

On $(-\infty,1)$, $(1,\infty)$ f is defined by formula, hence cts.

At 1, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x^3 - c) = 2\cdot 1^3 - c = 2-c$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (cx^2+3) = c\cdot 1^2 + 3 = c+3 = f(1)$

so f is cts at 1 if $2-c = c+3$, i.e. if $\boxed{c = -\frac{1}{2}}$

(3) Where are the following functions continuous?

$f(x) = \frac{1}{\sqrt{7-x^2}}$ $\quad(-\sqrt{7}, +\sqrt{7})$ (need $7-x^2>0$)

$g(x) = \frac{x^2+2x+1}{2+\cos x}$ $\quad\mathbb{R}$: for all x $2+\cos x \geq 2-1 > 0$

(or $\cos x \neq -2$ for all x)

$h(x) = \frac{2+\cos x}{x^2+2x+1} = \frac{2+\cos x}{(x+1)^2}$ $\quad(-\infty,-1)\cup(-1,\infty)$

$k(x) = \log(\sin x)$ where $\sin x > 0$

$(0,\pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \cup \ldots$

$= \bigcup_{k\in\mathbb{Z}} (2\pi k, 2\pi k + \pi)$ $\quad$ (Zahlen) = "numbers")

(4) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x\to 3}(xf(x) + g(x)) = 1$. Find $f(3)$.

$$1 = \lim_{x\to 3}(xf(x) + g(x)) \underset{\substack{\uparrow \\ \text{limit} \\ \text{laws}}}{=} \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} f(x)\right) + \left(\lim_{x\to 3} g(x)\right) =$$

$$\underset{\substack{\uparrow \\ f,g \text{ are cts at } 3}}{=} 3 \cdot f(3) + g(3) = 3 \cdot f(3) + 2 \quad \text{so } \boxed{f(3) = -\frac{1}{3}}.$$

2. The Intermediate Value Theorem

Theorem. *Let $f(x)$ be continuous for $a \leq x \leq b$. Then $f(x)$ takes every value between $f(a), f(b)$.*

(5) Show that $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \leq x \leq 1$.

Aside: The equation $t^2=1$ has two solutions (± 1), but $\sqrt{1}$ is defined to be the non-negative one