2. Limit laws (14/9/2021)

Goals.

(1) Limits, again
(2) Piecewise-defined functions
(3) Existence and nonexistence of limits: blowup
(4) Limit laws
(5) The squeeze/sandwich theorem

Last Time.

Note: No class Thursday

Limiting processes

$\rightsquigarrow$ limits of functions/expressions:

"the value the function would like to have"

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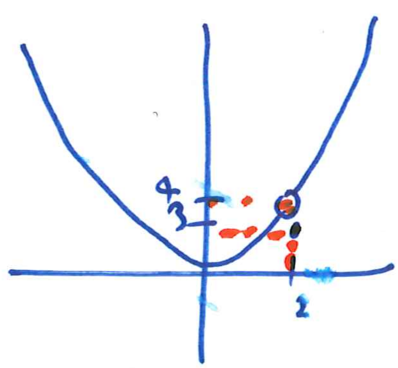
(1) Example: let $f(x) = \begin{cases} x^2 & x \neq 2 \\ 3 & x = 2 \end{cases}$



As $x \to 2$, $f(x) \to 4$

(if x close to 2, x^2 is close to 4)

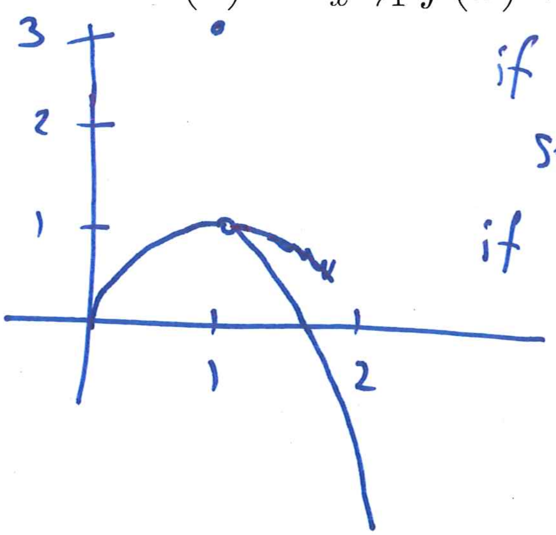
Worksheet 1(a), 1(b)

Math 100 – WORKSHEET 2
LIMIT LAWS

1. Existence of limits and blowup

(1) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.



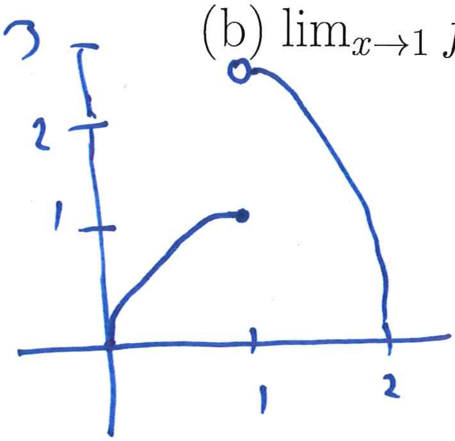
if $x<1$, $f(x)=\sqrt{x}$

So $f(x) = \sqrt{x} \xrightarrow[x\to 1^-]{} \sqrt{1} = 1$

if $x>1$, $f(x) = 2-x^2 \xrightarrow[x\to 1^+]{} 2-1^2 = 1$

So $\lim_{x\to 1} f(x) = 1$.

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.



$\lim_{x\to 1^-} f(x) = 1$ (same as above)

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4-x^2 = 4-1^2 = 3$

if $x>1$, $f(x) = 4-x^2$

Date: 14/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

since $1 \neq 3$, the limit does not exist. (DNE)

<u>Remarks:</u> (1) $\lim_{x \to a} f(x)$ not influenced by $f(a)$, doesn't have to equal it.

(2) we can define functions by different formulas at different points

(3) might need to test limits on left and right separately.

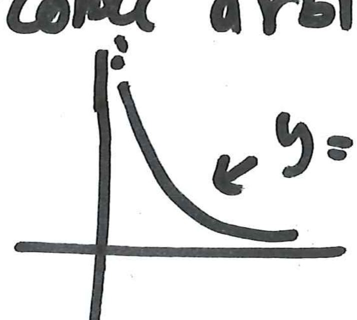
(if $\lim_{x \to a} f(x)$ does not exist we say f is discontinuous at a)

Why would limit not exist!

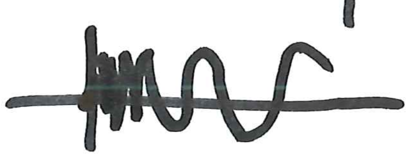
(1) <u>jump</u> : limits on left & right exist, but disagree.

(2) <u>blowup</u> : values become arbitrarily large.

Example: $\lim_{x \to 0} \frac{1}{x}$.



(3) oscillation



$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x>0 \\ 0 & x=0 \end{cases}$$

(2) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

$\left[\frac{3-3}{3^2+3-12} \text{ "=" } \frac{0}{0}\right]$ if $x \neq 3$, $f(x) = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4} \xrightarrow[x\to 3]{} \frac{1}{7}$

<u>if</u> expression is badly-behaved, need to think

(b) What about $\lim_{x\to 2} f(x)$?

$$\lim_{x\to 2} f(x) = \frac{\lim_{x\to 2}(x-3)}{\lim_{x\to 2}(x^2+x-12)} = \frac{2-3}{2^2+2-12} = \frac{-1}{-6} = \frac{1}{6}$$

if f is well-behaved at x, no problem

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challenge: suppose $\lim_{x\to 5} f(x) = 7$, $\lim_{x\to 5} g(x) = 8$, $\lim_{x\to 5} h(x) = 10$.

what is $\lim_{x\to 5} \frac{f(x)+g(x)}{f(x)\cdot g(x) + h(x)}$?

$$\lim_{x\to 5} \frac{f(x)+g(x)}{f(x)\cdot g(x)+h(x)} = \frac{(\lim_{x\to 5} f(x)) + (\lim_{x\to 5} g(x))}{(\ )(\ ) + \lim_{x\to 5} h(x)} = \frac{7+8}{7\cdot 8+10}$$

$= \frac{15}{66} = \frac{5}{22}$

2. Limit Laws

Fact. *Limits respect arithmetic operations and standard functions (e^x, sin, cos, log, ...)* **as long as everything is well-defined.**

(beware especially of division by zero)

(4) Evaluate using the limit laws:

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1} =$ $\frac{2+1}{4\cdot 2^2-1} = \frac{3}{15} = \frac{1}{5}$

(or $\lim_{x\to 2} \frac{x+1}{4x^2+1} = \frac{(\lim_{x\to 2} x)+1}{4(\lim_{x\to 2} x)^2 - 1} = \frac{2+1}{4\cdot 2^2 - 1} = \frac{3}{15} = \frac{1}{5}$

(b) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} =$

trick: $\sqrt{a}-\sqrt{b} = (\sqrt{a}-\sqrt{b})\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

$$= \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

(5) Evaluate:

(a) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$.

$$\frac{\sqrt{4+x}-2}{x} = \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \frac{(4+x)-4}{x\cdot(\sqrt{4+x}+2)} =$$

$$= \frac{\cancel{x}}{\cancel{x}(\sqrt{4+x}+2)} \underset{x\neq 0}{=} \frac{1}{\sqrt{4+x}+2} \xrightarrow[x\to 0]{} \frac{1}{4}$$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$.

Question:

What about $\lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1$

$$\lim_{x\to \pi} \sin\left(\frac{1}{x}\right) = \sin\left(\frac{1}{\pi}\right)$$

Worksheet (2)

Lesson: ok to work with unknown functions

On tests: every question tests some knowledge, answer should demonstrate that.

$\Leftrightarrow$ Must explain WHY ~~you~~ the answer is correct.

Worksheet (4), (5)
