6. The Derivative (28/9/2021)

Goals.

(1) The derivative at a point
(2) The derivative as a function
(3) Power laws and polynomials

Last Time.

IVT: If f is cts on $[a,b]$, if $f(a), f(b)$ have opposite ~~signs~~ signs then have c, $a<c<b$, s.t. $f(c)=0$

Used IVT to find solutions to equations

Often, have to choose a,b ourselves, then can use asymptotics.

Def: Let f be defined near $a \in \mathbb{R}$

The derivative of f at a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

provided the limit exists

Math 100 – WORKSHEET 6
THE DERIVATIVE

1. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

(1) Find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

$$f'(3) = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h\to 0} \frac{6h + h^2}{h} =$$

$$= \lim_{h\to 0} (6+h) = 6.$$

(b) $f(x) = \frac{1}{x}$, any $a \neq 0$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right)$$

$$= \lim_{h\to 0} \frac{-h}{h\cdot a \cdot a+h} = + \lim_{h\to 0} \frac{(-1)}{a(a+h)} = -\frac{1}{a^2}.$$

Date: 28/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

(2) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h) - \cos 5}{h}$ $= f'(5)$

$\lim_{\cancel{h} x \to 0} \frac{\sin x}{x}$ $= \lim_{x\to 0} \frac{\sin x - 0}{x}$

$= \lim_{x\to 0} \frac{\sin x - \sin 0}{x} = (\sin x)'(0)$

$f(x) = \cos x$

$a = 5$

Worksheet (1) (a), (b), (2), (3)

Language + notation

Noun: "derivative" of f at a

$\Rightarrow$ derivative function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Verb: to differentiate

Adj: differentiable = has a derivative (= limit exists)

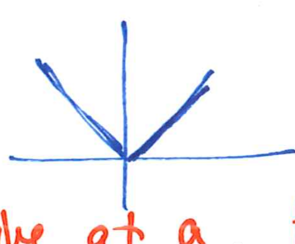
Write. $f'(x)$, or $f'(a)$,

or $\frac{df}{dx}$ (also Df, $D_x f$, ..)

if think of graph $y = f(x)$, write $\frac{dy}{dx}$.

Fact: If f is differentiable at a, f is also continuous at a

But $f(x) = |x|$



cts, not diff at $x = 0$

$f'(a)$ = derivative at a, $f'(x)$ = derivative at x

$f'(t)$ = " " t

f' = derivative function

Power law: function x^a

Fact: $\frac{d(x^a)}{dx} = a x^{a-1}$.

Worksheet (4), (5)

(3) (Final, 2015) Is the function

$$f(x) = \begin{cases} \sqrt{1+x^2} - 1 & x \le 0 \\ x^2 \cos \frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?

$f(0) = \sqrt{1-0} - 1 = 0$

$f'(0) \overset{?}{=} \lim_{x\to 0} \frac{f(x)-f(0)}{x}$

$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{\sqrt{1+x^2}-1}{x} = \lim_{x\to 0} \frac{(1+x^2)-1}{x} \cdot \frac{1}{\sqrt{1+x^2}+1} = \lim_{x\to 0} \frac{x}{\sqrt{1+x^2}+1} = 0$

$\parallel$

$\left(\sqrt{1+x^2}-1\right)'(0) = \left[\frac{2x}{2\sqrt{x^2+1}}\right]_{x=0} = 0$

$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2 \cos\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^+} x \cos\left(\frac{1}{x}\right) = 0$ by squeeze thm

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$\Downarrow$

$\underset{\substack{x\to 0 \\ 0}}{-x} \le x\cos\left(\frac{1}{x}\right) \le \underset{\substack{x\to 0 \\ 0}}{x}$

2. Linear combinations; power laws

(4) Let $g(y) = Ay^{5/2} + y^2$. Suppose that $g'(4) = 0$. What is A?

$\frac{dg}{dy} = g'(y) = A \cdot \frac{5}{2} \cdot y^{\frac{5}{2}-1} + 2y = \frac{5A}{2} \cdot y^{3/2} + 2y$

so $0 = g'(4) = \frac{5A}{2} \cdot 4^{3/2} + 2\cdot 4 = 20A + 8$ so $\boxed{A = -\frac{8}{20} = \frac{-2}{5}}$.

(5) Find the *second* derivative of $5t + 3\sqrt{t}$

if $y = 5t + 3\sqrt{t}$, then $y' = 5 + 3\cdot\frac{1}{2}\cdot t^{-1/2} = 5 + \frac{3}{2\sqrt{t}}$.

so $y'' = 0 + \frac{3}{2}\cdot\left(-\frac{1}{2}\right) t^{-3/2} = -\frac{3}{4} t^{-3/2}$.

constant doesn't change

(6) Differentiate $f(x) = \frac{5x^3-2x+1}{\sqrt{x}}$. $= \frac{5x^3}{\sqrt{x}} - \frac{2x}{\sqrt{x}} + \frac{1}{\sqrt{x}} =$

$= 5x^{5/2} - 2 \cdot x^{1/2} + x^{-1/2}$

(7) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

$f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

So line is $y = \frac{1}{4}(x-4)+2$

$\iff$ $y - 2 = \frac{1}{4}(x-4)$

$\iff$ $y = \frac{1}{4}x + 1$