# 10. Logarithmic and Implicit differentiation (14/10/2021)

Goals.

(1) Differentiation involving logarithms
(2) Implicit differentiation

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Last Time. **Chain rule** $f(g(x))' = f'(g(x))g'(x)$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

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Special case: $f(g(x)) = x$ ie. $g = f^{-1}$

we set $\frac{df}{dx} \cdot \frac{dx}{df} = 1$ ("inverse fcn rule")

$$f'(x) \cdot (f^{-1})'(f^{-1}(x)) = 1$$

**Example:** $\log = (\exp)^{-1}$ so $\frac{d(\log x)}{dx} = \frac{1}{e^{\log x}} = \frac{1}{x}$

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One point on inverse functions:

We focused on $f^{-1}$ as a function with a formula.
Recall that it also makes sense point-by-point.

I.e. $y = f(x)$ often means: "given $x$, what is $y$?"

So $x = f^{-1}(y)$ asks: "given $y$, what $x$ value produced it?"

$\Longleftrightarrow$ treats $y = f(x)$ as an equation for $x$.

Example: Say $f(x) = 2 + 2x + \sin x$

What is $f^{-1}(2+2\pi)$?

This is the $x$-value s.t. $f(x) = 2 + 2\pi$.

We guess: see $x = \pi$ works

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A process grows / decays exponentially if it's described by

$$F(t) = C \cdot e^{kt} = C q^t \quad (q = e^k)$$

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**Worksheet (1), (2)**

Math 100 – WORKSHEET 10
LOGARITHMIC AND IMPLICIT DIFFERENTIATION

# 1. Review of Logarithms

(1) $\log\left(e^{10}\right) = 10$ $\qquad$ $\log(2^{100}) = 100(\log 2)$

(2) A variant on *Moore's Law* states that computing power doubles every 18 months. Suppose computers today can do $N_0$ operations per second.

(a) Write a formula predicting the future:

- Computers $t$ years from now will be able to do $N(t)$ operations per second where

$$N(t) = N_0 \cdot 2^{t/1.5} = N_0 \cdot e^{\frac{\log 2}{1.5} t}$$

After $t$ years have $\frac{t}{1.5}$ doublings

so $N(t) = N_0 \cdot 2^{t/1.5}$

Date: 14/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

# 2. Differentiation

$$\boxed{(\log x)' = \frac{1}{x}}$$

chain rule

(1) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$

$$= \frac{d(\log x + \log a)}{dx} = \frac{1}{x}$$

or $= \frac{1}{ax} \cdot a = \frac{1}{x}$

chain rule

$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) = \frac{1}{t^2+3t} \cdot (2t+3)$

$$= \frac{2t+3}{t^2+3t}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$

$$= 2x \log(1+x^2) + \frac{x^2 \cdot 2x}{1+x^2}$$

pdt rule

chain rule

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$

$$- \frac{1}{(\log(2+\sin r))^2} \cdot \frac{\cos r}{(2+\sin r)}$$

quotient rule

chain rule

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

# Logarithmic differentiation

$\log(xy) = \log x + \log y$ addition is easier

Use for diff:

say $f(x) = \dfrac{x^2+1}{x+1} \cdot \dfrac{\cos x + 7}{1+e^x}$ what is $f'$?

then $\log f = \log(x^2+1) + \log(7+\cos x)$

$\qquad - \log(1+x) - \log(1+e^x)$

diff both sides

$$(\log f)' = \frac{f'}{f} = \frac{2x}{x^2+1} + \frac{-\sin x}{7+\cos x} - \frac{1}{1+x} - \frac{e^x}{1+e^x}$$

$$\text{so } f' = \left(\frac{x^2+1}{x+1} \cdot \frac{\cos x + 7}{1+e^x}\right)\left(\frac{2x}{x^2+1} - \frac{\sin x}{7+\cos x} - \frac{1}{1+x} - \frac{e^x}{1+e^x}\right)$$

(2) (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

$\frac{1}{\sqrt{x^3+3}} = (x^3+3)^{-\frac{1}{2}}$

$\log y = \log(x^2+1) + \log(\sin x) - \frac{1}{2}\log(x^3+3) + \cos x$

so $\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$

so $y' = \left((x^2+1)\sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}\right)\left(\frac{2x}{x^2+1} + \frac{1}{\tan x} - \frac{3x^2}{2(x^3+3)} - \sin x\right)$.

don't forget to solve for $y$!

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<u>Before</u>: $(x^a)' = a x^{a-1}$, $(a^x)' = (\log a) \cdot a^x$.

What if <u>both</u> base <u>and</u> exponent are changing?

<u>Today</u>: hit it with log

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(e.g. $\log(a^x) = x \log a$ so $(\log(a^x))' = \log a$

so indeed $(a^x)' = a^x \cdot (\log(a^x))'$.

(3) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) $x^x$

$\log(x^x) = x\log x$ so $(x^x)' = x^x \cdot (x\log x)' =$

$= x^x(\log x + 1)$

$1 \cdot \log x + x \cdot \frac{1}{x}$

(aside: find $f$ s.t. $f'(x) = \log x$)

(b) $(\log x)^{\cos x}$

$((\log x)^{\cos x})' = (\log x)^{\cos x} \cdot (\cos x \cdot \log(\log x))' =$

using logarithmic differentiation

$= (\log x)^{\cos x}(-\sin x \cdot \log\log x + \cos x\,(\log\log x)')$

$= (\log x)^{\cos x}\left(-\sin x \log\log x + \cos x \frac{1}{\log x} \cdot \frac{1}{x}\right)$

(c) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ only.

$\log y = \log(x^{\log x}) = \underset{\text{exponent}}{(\log x)} \cdot \underset{\text{log of base}}{(\log x)} = (\log x)^2$ so by chain rule

~~deb~~ $\frac{y'}{y} = (\log y)' = 2\log x \cdot \frac{1}{x}$ so

$y' = 2\log x \cdot \frac{1}{x} \cdot y = 2\log x \cdot \frac{1}{x} \cdot x^{\log x} = 2\log x \cdot x^{\log x - 1}$

# Implicit diff

Suppose we have a relation between x an y:

$$5x^2 + 3y^2 = 8$$

$$\left[ \Leftrightarrow y = \pm\sqrt{\frac{8-5x^2}{3}} \right]$$

Can differentiate the relation (along the curve) & solve for y':

$$\frac{d}{dx}(5x^2) = 10x$$

$$\frac{d}{dx}(3y^2) = 6y \cdot \frac{dy}{dx}$$

$$10x + 6y \cdot y' = 0$$

so along the curve $y' = -\frac{10x}{6y} = -\frac{5x}{3y}$

eg. if $x = 1$, $y = 1$, $y' = -5/3$

if have point (1,1) on curve, can find the slope at that point without solving $y = f(x)$ first

## 3. Implicit Differentiation

(1) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.

(2) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

(3) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

diff wrt $x$: $y' + \cos y - x(\sin y)\cdot y' = -\sin x$

set $x=0$, $y=1$ get $y' + \cos 1 = 0$ so $y' = -\cos 1$.