11. Inverse Trig; Related Rates (19/10/2021)

Goals.

(1) Evaluating inverse trig functions
(2) Differentiating inverse trig functions
(3) Related Rates

Last Time. $\frac{d(\log x)}{dx} = \frac{1}{x} \Rightarrow (\log f)' = \frac{f'}{f}$
$\Rightarrow f' = f \cdot (\log f)'$ useful since log converts products to sums, powers to products.

Can differentiate a relation between x, y wrt x to get a relation between x, y, y'!

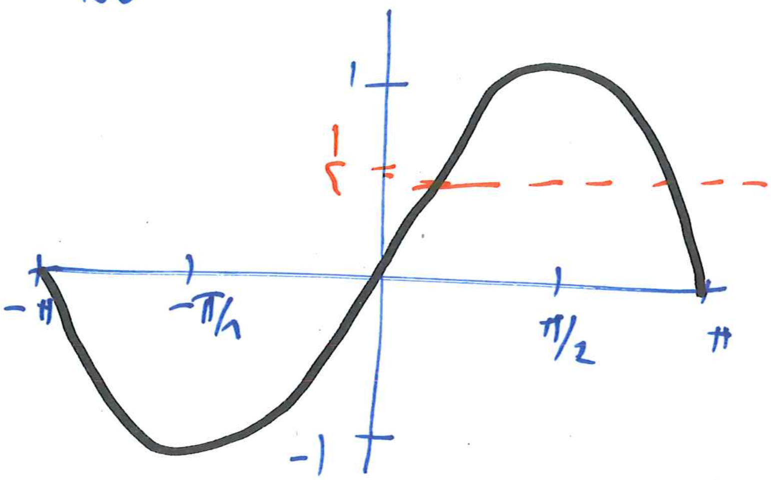
(e.g. solve for y' in terms of x, y)

Example: relation $x^y = y^x$. problem: x^y not a power law x^a, not an exponential a^y. Hit with log, get new relation, diff that

Inverse trig

Does $\sin\theta$, $\theta \in \mathbb{R}$ have an inverse?

No



On $[-\pi/2, \pi/2]$ $\sin\theta$ takes every value in $[-1,1]$ exactly once, so has an inverse there.

Def: $\arcsin x = \theta$ if $\sin\theta = x$ and $-\pi/2 \le \theta \le \pi/2$

$\arccos x = \theta$ if $\cos\theta = x$ and $0 \le \theta \le \pi$

$\arctan x = \theta$ if $\tan\theta = x$ and $-\pi/2 < \theta < \pi/2$

Warning: $\arcsin(\sin\theta)$ need not be θ

just like $\sqrt{(-5)^2} = 5$

Math 100 – WORKSHEET 11
INVERSE TRIG FUNCTIONS; RELATED RATES

1. Inverse trig functions

(1) Evaluation

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\pi/6$

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want θ s.t. $\sin(\theta) = \sin\left(\frac{31\pi}{11}\right)$ & $\theta \in [-\pi/2, \pi/2]$

$\frac{31}{11} \approx 3$ so $\frac{31\pi}{11} \notin [-\pi/2, \pi/2]$: instead

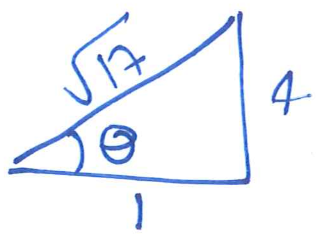
$\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9}{11}\pi\right) = \sin\left(\pi - \frac{9}{11}\pi\right) = \sin\left(\frac{2}{11}\pi\right)$

so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right) = \frac{2}{11}\pi$.

$\sin(\pi - \theta) = \sin\theta$

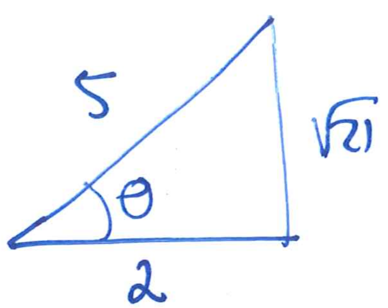
(b) (Final 2015) Simplify $\sin(\arctan 4)$

let $\theta = \arctan 4$



so $\sin\theta = \frac{4}{\sqrt{17}}$.

(c) Find $\tan(\arccos(0.4))$



so $\tan(\arccos(0.4)) = \frac{\sqrt{21}}{2}$.

(choose two sides so θ has the right value, find 3rd using Pythagoras, read off trig function)

Differentiating arc sin etc

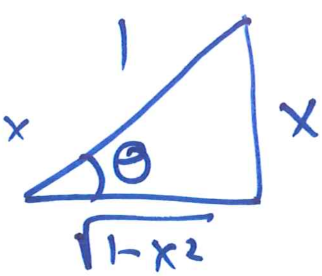
let $\theta = \arcsin x$, then $x = \sin\theta$

so $1 = \frac{dx}{dx} = \frac{d(\sin\theta)}{dx} = \cos\theta \cdot \frac{d\theta}{dx}$

implicit diff a.k.a. chain rule

so $\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$

$\sin\theta = x$



$\cos\theta = \sqrt{1-x^2}$

simplify $\cos(\arcsin x)$

$(\cos(\arcsin x) \geq 0$ for all $x)$

Similarly: $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin x$

$\Rightarrow \boxed{(\arccos x)' = -(\arcsin x)'}$

$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$

$\boxed{(\arctan x)' = \frac{1}{1+x^2}}$

(2) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$ $= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$

$\frac{d(2x)}{dx}$

(b) Find the line tangent to $y = \sqrt{1+(\arctan(x))^2}$ at the point where $x = 1$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+(\arctan x)^2}} \cdot 2(\arctan(x)) \cdot \frac{1}{1+x^2}$$

at $X=1$ $\arctan X = \arctan 1 = \frac{\pi}{4}$.

$(=45^\circ = \frac{\pi}{4})$

(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?

if position is $x(t)$, velocity is $v(t) = \frac{dx}{dt}$, acceleration is $a(t) = \frac{dv}{dt}$.

2. Velocity and Acceleration

(3) A particle's position is given by $f(t) = t + 6e^{-t/3}$.

(a) Find the velocity at time t, and specifically at $t = 3$.

$v(t) = \frac{df}{dt}(t) = 1 - 2e^{-t/3}$

$v(3) = 1 - \frac{2}{e}$

(b) When is the particle moving to the right? to the left?

moving right $\iff$ $v(t)$ so ire. $1 > 2e^{-t/3} \iff e^{t/3} > 2$

$\iff t > 3\log 2.$

(moving left if $t < 3\log 2$)

(c) When is the particle accelerating? decelerating?

$a(t) = \frac{dv}{dt}(t) = \frac{2}{3}e^{-t/3}$

determine when a, v have same or opposite signs

Related Rates

Say we have a relation between two quantities: $x^2+y^2=1$

$$P = T\cdot(\rho\cdot k)$$

pressure, temp, density, gas constant

imagine **both** quantities change with time can still take $\frac{d}{dt}$, get relation involving $x, y, \frac{dx}{dt}, \frac{dy}{dt}$.

Eg: if $x^2+y^2=1$ then $2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=0$.

Given any two of $x, y, \frac{dx}{dt}, \frac{dy}{dt}$, can solve for other two

3. Related Rates

(6) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $(1, \sqrt{3})$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.

diff wrt t: $2y\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 2\frac{dx}{dt}$

set $x=1$, $y=\sqrt{3}$, $\frac{dy}{dt}=1$, get $2\sqrt{3}\cdot 1 = 3\frac{dx}{dt} + 2\frac{dx}{dt}$

so $\frac{dx}{dt} = \frac{2\sqrt{3}}{5}$.

Alternative solution: diff wrt y:

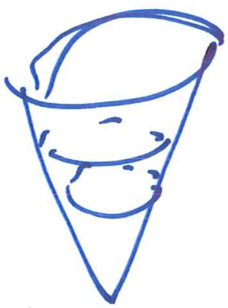
$$2y = 3x^2\frac{dx}{dy} + 2\frac{dx}{dy} \text{ so}$$

$$\frac{dx}{dy} = \frac{2y}{3x^2+2} \quad \left(= \frac{2\sqrt{3}}{5} \text{ at } \begin{matrix} y=\sqrt{3} \\ x=1 \end{matrix}\right)$$

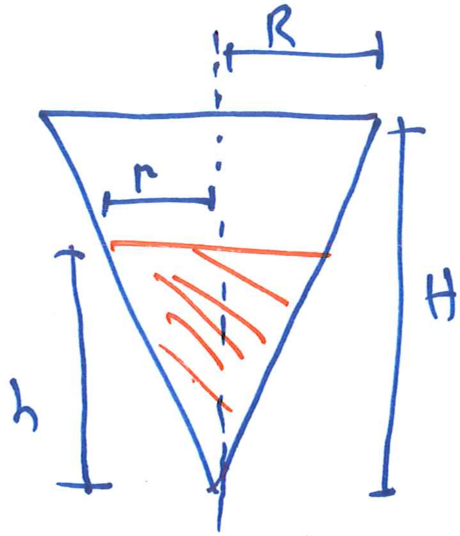
then $\frac{dx}{dt} = \frac{dx}{dy}\cdot\frac{dy}{dt} = \frac{2\sqrt{3}}{5}$

(7) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?



drawing



names

H: height of cone

R = radius " "

V: volume of water

h = height of water

r = radius of (top) of water

(b) The drain is unclogged and water begins to clear at the rate of $\frac{\pi}{4}\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?