14. Taylor expansion (28/10/2021)

Goals.

(1) Higher-order approximation
(2) Combining expansions

Last Time. MVT: $f(x) = f(a) + f'(c)(x-a)$
for some c between a, x.

linear approx: $f(x) \approx f(a) + f'(a)(x-a)$

(MVT also: $\frac{f(x)-f(a)}{x-a} = f'(c)$
i.e. there is c s.t. slope at c
= average slope
between a, x)

Midterm material: Week 1-6 of course syllabus

---

worksheet (1)

---

eg $f(x) = \log x$, $a=1$. $f'(x) = \frac{1}{x}$ so $f(1) = \log 1 = 0$
$f'(1) = \frac{1}{1} = 1$

so $f\left(\frac{4}{3}\right) \approx f(1) + f'(1)\left(\frac{4}{3}-1\right) = 0 + 1\left(\frac{4}{3}-1\right) = \frac{1}{3}$

$f\left(\frac{2}{3}\right) \approx f(1) + f'(1)\left(\frac{2}{3}-1\right) = 0 + 1\left(\frac{2}{3}-1\right) = -\frac{1}{3}$

Math 100 – WORKSHEET 14
TAYLOR EXPANSION

1. Taylor approximation

(1) (Review) Use linear approximations to estimate:

(a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.

let $f(x) = \log(1+x)$, work about $a=0$ (or $f(x) = \log x$ work about $a=1$)

~~f(x) =~~ $f'(x) = \frac{1}{1+x}$ so $f(0) = \log 1 = 0$, $f'(0) = \frac{1}{1+0} = 1$

so ~~f(1/3)~~ $\log\left(\frac{4}{3}\right) = f\left(\frac{1}{3}\right) \approx 0 + 1\cdot\left(\frac{1}{3} - 0\right) = \frac{1}{3}$

$\log\left(\frac{2}{3}\right) = \log\left(1 - \frac{1}{3}\right) = f(-1/3) \approx -\frac{1}{3}$

$$\boxed{\log 2 = \log\frac{4/3}{2/3} = \log\frac{4}{3} - \log\frac{2}{3} \approx \frac{1}{3} - \left(-\frac{1}{3}\right) = 2/3}$$

(b) $\sin 0.1$ and $\cos 0.1$.

$\sin 0 = 0$ $\cos 0 = 1$

$(\sin')(0) = 1$ $(\cos')(0) = 0$

$\sin 0.1 \approx 0 + 1\cdot(0.1) = 0.1$

$\cos 0.1 \approx 1 + 0\cdot(0.1) = 1$ ← derivative was zero

Date: 28/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(a) $f(x) = e^x$, $f'(x) = e^x$, $f^{(2)}(x) = e^x$, ...

so $f(0) = 1$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 1$, $f^{(3)}(0) = 1$, ...

(b) $f(0) = 1$, take $T_0(x) = 1$ (guess "e^x is close to $e^0 = 1$")

(c) $f(0) = 1$, $f'(0) = 1$ $T_0(x) = 1 + x$ (if not sure, try $T_0(x) = 1 + ax$, then $T_0'(x) = a$ so need $a = 1$)

$= 1 + 1 \cdot x$ (linear approx)

(d) $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, take $T_2(x) = 1 + x + \frac{x^2}{2}$

(try $T_2(x) = 1 + x + a_2 x^2$ $T_2''(x) = 2a_2$) $T_2''' = 0$

(e) try $T_3(x) = T_2(x) + a_3 x^3$ $T_3'''(x) = 0 + 6a_3$

choose $a_3 = \frac{1}{6}$, get $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

Takeaway: to make a polynomial $T_n(x)$ matching $f(0), f'(0), \dots, f^{(n)}(0)$

we took $T_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

to find a_k we diff. k times, we found

$$a_k = \frac{1}{1\cdot 2\cdot 3\cdots k}\cdot f^{(k)}(0)$$

$$a_0 = f(0),\quad a_1 = \frac{f'(0)}{1},\quad a_2 = \frac{f''(0)}{1\cdot 2},\quad a_3 = \frac{f^{(3)}(0)}{1\cdot 2\cdot 3},\ \dots$$

Worksheet (3)

Formula: (Memorize) If we expand f about a then the coefficient of $(x-a)^k$ is

$$c_k = \frac{f^{(k)}(a)}{k!}$$

$$k! = 1\cdot 2\cdot 3\cdots k$$

Call $T_n(x) = c_0 + c_1(x-a) + c_2(x-a) + \dots + c_n(x-a)^n$

$$= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

then n'th order Taylor expansion/polynomial of f about $x=a$.

(if $a=0$ call it also the MacLaurin's expansion)

(3) Do the same with $f(x) = \ln x$ about $x = 1$.

$f(x) = \log x \qquad f'(x) = \frac{1}{x}, \quad f^{(2)}(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}$

$f(1) = 0 \qquad f'(1) = 1 \qquad f^{(2)}(1) = -1 \qquad f^{(3)}(1) = 2$

$T_0(x) = 0$

$T_1(x) = (x-1)$

$T_2(x) = (x-1) + \left(-\frac{1}{2}\right)(x-1)^2$

$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

$2 \cdot \left(-\frac{1}{2}\right) = -1 \qquad 6 \cdot \frac{1}{3} = 2$

$T_2(x) = (x-1) + a_2(x-1)^2$

$T_2''(x) = 0 + 2a_2$

$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + a_3(x-1)^3$

$T_3'''(x) = 0 + 0 + 6a_3$

Takeaway: if expanding about $a = 1$ will have powers of $x - 1$.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$, $\left(\frac{1}{1-x}\right)^{(2)} = \frac{2}{(1-x)^3}$, $\left(\frac{1}{1-x}\right)^{(3)} = \frac{6}{(1-x)^4}$

~~$\left(\frac{1}{1-x}\right)\Big|_{x=0}$~~ let $f(x) = \frac{1}{1-x}$ $\left(\frac{1}{1-x}\right)^{(4)} = \frac{24}{(1-x)^5}$

so ~~$\left(\frac{1}{1-x}\right)'(0) =$~~ $f(0) = 1$, $f^{(1)}(0) = 1$ $f^{(2)}(0) = 2$ $f^{(3)}(0) = 6$

$f^{(4)}(0) = 24$

$$T_4(x) = 1 + 1\cdot x + \frac{1}{2}\cdot 2\cdot x^2 + \frac{1}{6}\cdot 6\cdot x^3 + \frac{1}{24}\cdot 24\cdot x^4$$

$$= 1 + x + x^2 + x^3 + x^4$$

(5) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

$g(x) = \cos x \quad g^{(1)}(x) = -\sin x \quad g^{(2)}(x) = -\cos x \quad g^{(3)}(x) = \sin x$

$g^{(4)}(x) = \cos x \quad g^{(5)}(x) = -\sin x \quad . \; . \;$ repeats

$g(0) = 1,\; g^{(1)}(0) = 0,\; g^{(2)}(0) = -1,\; g^{(3)}(0) = 0,$ repeat

So expansion: $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40{,}320}x^8 + .$

no linear term: $g^{(1)}(0) = 0$

no cubic term $g^{(3)}(0) = 0$

Common error

Expansion is <u>not</u> $1 + (-\sin x)x + .$

Coeff is $\frac{g'(0)}{1} \neq g'(x)$

E.g. $T_3(x) = 1 - \frac{1}{2}x^2$

$T_3(0.1) = 1 - \frac{1}{200}$

$\left(\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - . \; . \right)$

to 4^{th} order

$\cos(0.1) \approx 1 - \frac{1}{200} + \frac{1}{240{,}000}$

---

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

formula: $C_k = \dfrac{f^{(k)}(a)}{k!}$ here $12 = C_2 = \dfrac{f''(3)}{2}$

so $f''(3) = 24$.

2. New from old

(7) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x=0$.

expand $\sin x$ about 0: get $x - \frac{x^3}{6}$

($\sin 0 = 0$, $\cos 0 = 1$, $-\sin 0 = 0$, $-\cos 0 = -1$)

so expansion of $(x+1)\sin x$ is that of

$$(x+1)\left(x - \frac{x^3}{6}\right) = x - \frac{x^3}{6} + x^2 - \cancel{\frac{x^4}{6}}$$

i.e. $x + x^2 - \frac{x^3}{6}$

(expand both fcns to 3^{rd} order, multiply, keep terms up to 3^{rd} order)