18. The shape of the graph (16/11/2021)

Goals.

(1) Midterm review
(2) Implications of MVT for the shape of the graph:
 (a) Increasing and decreasing functions
 (b) Concave and convex functions

Last Time. Optimization.

(1) give quatities names ; (2) use relations between them to make objective function (3) calculus ; (4) answer question

Question: How do we do calculus part?

Want absolute max/min : find them by looking for local max/min

absolute: over all domain

local: in a small region

Fact: if x_0 is a local ~~ex~~extremum then either $f'(x_0)=0$ or $f'(x_0)$ DNE.

~~Real quest~~ Absolute max is at such a pt or at endpoint.

Real question: How do we tell if a critical/singular point x_0 is a local max/min?

Midterm review: (1) What kinds of mistakes were common
(2) Practising checking our work

Midterm review

not a limit $\lim_{x\to-\infty}\sqrt{4x^2-3x}+2x+1=\lim_{x\to\infty}\sqrt{4x^2+3x}-2x+1=\lim_{x\to\infty}x\left(\sqrt{4-3/x}-2+\frac{1}{x}\right)$

can't take limit of part $\rightarrow =\lim_{x\to\infty}x\cdot 0=0$

1. Evaluate the following limits:

   (a) $\lim_{x\to-\infty}\sqrt{4x^2-3x}+2x+1$

$\lim_{x\to-\infty}\sqrt{4x^2-3x}+2x+1=\left(\sqrt{4x^2-3x}+2x+1\right)\cdot\frac{\sqrt{4x^2-3x}-2x-1}{\sqrt{4x^2-3x}-2x-1}=\frac{4x^2-3x-(2x^2-1)}{\sqrt{4x^2-3x}-2x+1}=$ $(x+1)^2$ -1

$\sqrt{x^2}=-x\rightarrow$ $=\frac{2x^2-3x-1}{-2x\sqrt{1-\frac{3}{4x}}-2x-1}=\lim_{x\to-\infty}\frac{x^2}{x}\cdot\frac{2-\frac{3}{x}-1/x^2}{2\sqrt{1-\frac{3}{4x}}-2-\frac{1}{x}}=\frac{2-0-0}{2\sqrt{1-0}-2}=\frac{2}{2-2}=\frac{2}{0}$ DNE.

disappeared?

   (b) $\lim_{t\to-3}\frac{2t+6}{\sqrt{t+4}-1}$

$\lim_{t\to-3}\frac{2t+6}{\sqrt{t+4}-1}=\lim_{t\to-3}\frac{2t+6}{\sqrt{t+4}-1}\cdot\frac{\sqrt{t+4}+1}{\sqrt{t+4}+1}=\lim_{t\to-3}\frac{(2t+6)(\sqrt{t+4}+1)}{t+4-1}$ $\xrightarrow[t\to3^+]{}\infty$, $\xrightarrow[t\to3^-]{}-\infty$ $\Rightarrow$ DNE

or $=\lim_{t\to-3}2\sqrt{t+4}+1=3$

notice: numerator also vanishes at $t=-3$.

2. Show that there is a number c such that $\tan(c)=c+1$.

Let $f(x)=\tan(x)-(x+1)$. Then $f(0)=1$, $f(\pi)=0-\pi+1=-(\pi-1)<0$

Thus f has a zero between $0,\pi$. no mention of IVT

(1) didn't check continuity; (2) $\tan(x)$ blows up at $\frac{\pi}{2}$; (3) no endgame

3. Differentiate

   (a) $(3+x)^{\frac{3}{x}}$ $(x>3)$ ← should be $3/x$

$\log(3+x)^{3/x}=\frac{\log 3}{\log x}\log(3+x)$ so $\left((3+x)^{3/x}\right)'=-\frac{\log 3}{(\log x)^2}\frac{1}{x}\log(3+x)+\frac{\log 3}{(3+x)\log x}$

chain rule

$f'=(f)\cdot(\log f)'$

   (b) $\sin x\cos(x^2+x)$

$\frac{d}{dx}(\sin x\cos(x^2+x))=-\cos x\sin(x^2+x)(2x+2)$ $(fg)'\neq f'g'$

5. A population of algae decays exponentially.

   (a) If the population falls by a factor of 3 every 30 days, find the time needed for the population to be divided by 2.

$N=N_0e^{-kt}$ $N(30)=\frac{2}{3}N_0$ so $\frac{2}{3}=e^{-k30}$ so $\frac{\log 2}{\log 3}=-k\cdot30$ so $k=-\frac{\log 3}{30\log 2}$

so $N(t)=\frac{1}{2}N_0$ when $\frac{1}{2}=e^{-kt}\Rightarrow\frac{1}{2}=e^{\frac{\log 3}{\log 2\cdot 30}t}\Rightarrow\log\frac{1}{2}=\frac{\log 3}{\log 2}\frac{t}{30}$

$\Rightarrow t=30\cdot\frac{\log\frac{1}{2}\log 2}{\log 3}$.

   (b) If the initial population is 100, what is the population after 10 days?

$N(10)=e^{\frac{\log 3}{\log 2}\cdot\frac{10}{30}}=e^{\frac{\log 3}{3\log 2}}$.

why is $N(10)>N(0)$?

Conclusions:

(1) Grading is about the process.

(2) Lots of algebra errors

(3) check your work!

look for mistakes — make sanity checks

English: "fall by $\frac{1}{5}$-th" is additive ($\frac{4}{5}$-th remain)

"fall by factor of 5" is multiplicative ($\frac{1}{5}$-th remains)

The shape of the graph

Say we have function f we want to understand

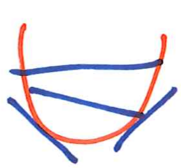
(0) can tell where $f(x) > 0$, $f(x) < 0$, $f(x) = 0$ + vertical, horizontal asymptotes } 0'th derivative info

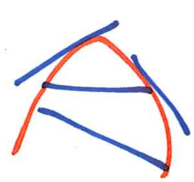
(1) can tell where $f'(x) > 0$, $f'(x) < 0$, $f'(x) = 0$, DNE ($\Rightarrow$ increasing/decreasing, local extrema) } 1'st derivative info

(2) can tell where $f''(x) > 0$, $f''(x) < 0$, $f''(x) = 0$

"concave up" — "concave down"

secant lines above graph tangent lines below





secant lines below graph tangent lines above

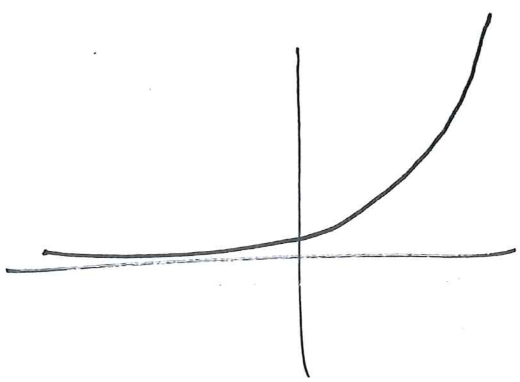
(2) For each of the following functions determine its domain, and where it is increasing or decreasing. Except in part (b) also determine where the function is concave up/down.

(a) $f(x) = e^x$

f always positive, $f'(x) = e^x$ always positive $\Rightarrow$ f is increasing.

$f''(x) = e^x$ so f is concave up

(also $\lim_{x \to -\infty} e^x = 0$)



(b) $f(x) = \frac{x-2}{1+x^2}$

(c) $f(x) = x \log x - 2x$

(d) $\frac{x^2-9}{x^2+3}$. You may use that $f'(x) = \frac{24x}{\left(x^2+3\right)^2}$ and that $f''(x) = 72\frac{1-x^2}{(x^2+3)^3}$.

f is positive on $(-\infty,-3) \cup (3,\infty)$, negative on $(-3,3)$, vanishes at ± 3.

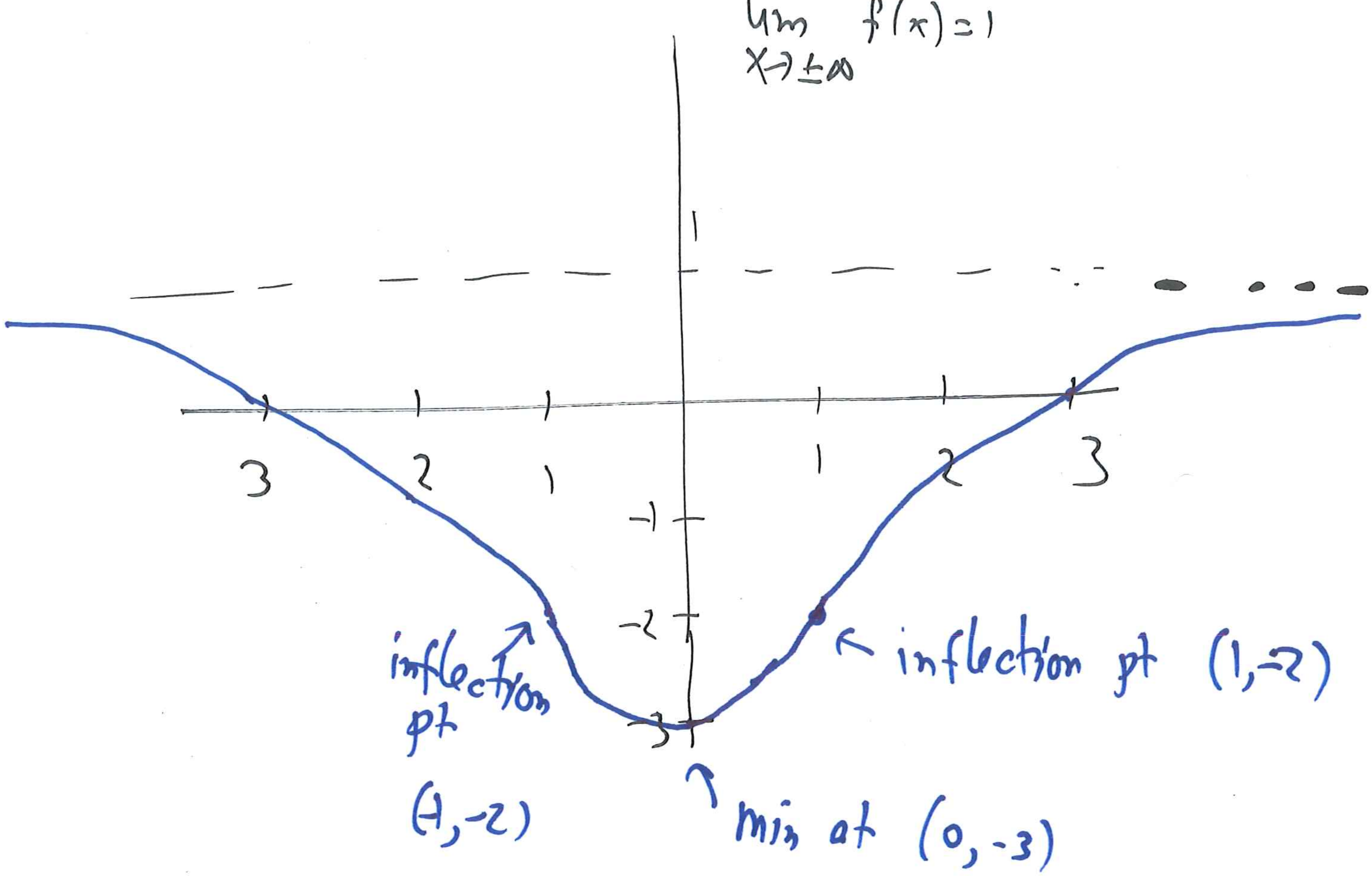
$f' > 0$ on $(0,\infty)$, negative on $(-\infty,0)$, vanishes at 0.

$\Rightarrow f(0) = -\frac{9}{3} = -3$ is a local min

$f''(x) > 0$ on $(-1,1)$, negative on $(-\infty,-1) \cup (1,\infty)$

So ± 1 are inflection points = points where concavity changes

x	$(-\infty,-3)$	-3	$(-3,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,3)$	3	$(3,\infty)$
f	$+$	0	$-$	-2	$-$	-3	$-$	-2	$-$	0	$+$
f'	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
f''	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$



$\lim_{x \to \pm\infty} f(x) = 1$
1
3
2
1
1
2
3
-1
-2
-3
inflection pt (-1,-2)
inflection pt (1,-2)
min at (0,-3)