19. Curve Sketching (18/11/2021)

Goals.

(1) Curve sketching protocol
(2) Examples from past exams

Last Time. Effect of the MVT on graphs:

(1) $f' > 0 \Rightarrow f$ is increasing ↗
$f' < 0 \Rightarrow f$ is decreasing ↘

(2) $f'' > 0 \Rightarrow f$ is concave up ∪
$f'' < 0 \Rightarrow f$ is concave down ∩

either can be used to identify local extrema

(0) can also look at values of f: where $f > 0$, $f < 0$, $f = 0$, .
$\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

If $f'(x_0) = 0$, $f'(x)$ positive to the right of x_0, negative to the left of x_0 ↘↗
then x_0 is a local minimum

If $f'(x_0) = 0$, $f''(x_0) < 0$, f is concave down near x_0 ∩
so x_0 is a local maximum

Example: let $f(x) = x^{2/3}(x-1)$

(0) f is defined on $\mathbb{R} = (-\infty, \infty)$ ($x^{1/3}$ is inverse to x^3 on $\mathbb{R}$)

$x^{2/3} = (x^{1/3})^2 \geq 0$, so $f(x) > 0$ if $x > 1$

$f(x) < 0$ if $x < 1$, $x \neq 0$

$f(x) = 0$ if $x = 1$ or $x = 0$

[for large $|x|$, $f(x) \sim x^{5/3}$]

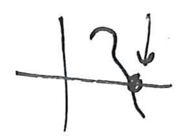
$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

(1) $f'(x) = \frac{2}{3} x^{-1/3}(x-1) + x^{2/3} = \frac{1}{3x^{1/3}}\left(2(x-1) + 3x\right)$

$= \dfrac{5x-2}{3x^{1/3}}$

point where f vanishes



or $f'(x) = \left(x^{5/3} - x^{2/3}\right)' = \frac{5}{3}x^{2/3} - \frac{2}{3}x^{-\frac{1}{3}} = \dfrac{5x-2}{3x^{1/3}}$.

See: singular pt at $x=0$, critical pt at $\frac{2}{5}$.

[if f' both >0 & <0 on $(0, \frac{2}{5})$ by IVT f' would vanish there, but it doesn't!]

f' is positive on $(-\infty, 0)$, negative on $(0, \frac{2}{5})$, positive on $(\frac{2}{5}, \infty)$

$\begin{pmatrix} 5x-2<0 \\ x^{1/3}<0 \end{pmatrix}$ $\begin{pmatrix} 5x-2<0 \\ x^{1/3}>0 \end{pmatrix}$ $\begin{pmatrix} 5x-2>0 \\ x^{1/3}>0 \end{pmatrix}$

(2) $f''(x) = \frac{5}{3} \cdot \frac{2}{3} x^{-1/3} - \frac{2}{3}\left(-\frac{1}{3}\right) x^{-4/3} = \dfrac{10x+2}{9x^{4/3}}$

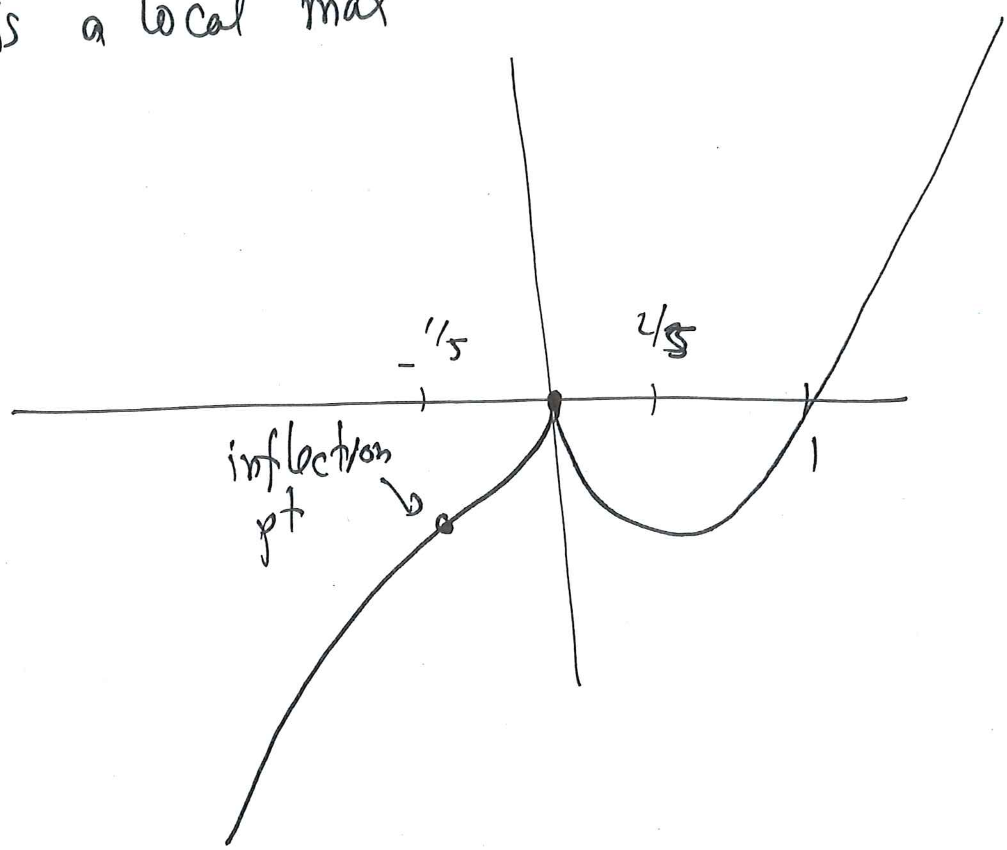
$f''(-\frac{1}{5}) = 0$, undef at $x=0$, $x^{4/3} > 0$ if $\neq 0$, so

$f''(x)$ is positive $(-\frac{1}{5}, 0) \cup (0, \infty)$, negative $(-\infty, -\frac{1}{5})$

x	$(-\infty, -\frac{1}{5})$	$-\frac{1}{5}$	$(-\frac{1}{5}, 0)$	0	$(0, \frac{2}{5})$	$\frac{2}{5}$	$(\frac{2}{5}, 1)$	1	$(1, \infty)$
f	$-$	$-$	$-$	0	$-$	$-$	$-$	0	$+$
f'	$+$	$+$	$+$	DNE	$-$	0	$+$	$+$	$+$
f''	$-$	0	$+$	DNE	$+$	$+$	$+$	$+$	$+$

$x_0 = \frac{2}{5}$ is a local min

$x_0 = 0$ is a local max



[16] **4.** Let $f(x) = x\sqrt{3-x}$.

(a) (2 marks) Find the domain of $f(x)$.

f is defined where $3-x \geq 0$
$\Leftrightarrow 3 \geq x$

Answer: $x \leq 3$ (or $[-\infty, 3]$)

(b) (4 marks) Determine the x-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.

$$f'(x) = \sqrt{3-x} + x\frac{1}{2\sqrt{3-x}}(-1) = \frac{2(3-x)-x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}} = \frac{3(2-x)}{2\sqrt{3-x}}$$

$f'(x)$ has same sign as $2-x$ so $f'(x) > 0$ on $(-\infty, 2)$
$f'(x) < 0$ on $(2, 3)$
$f'(2) = 0$

so f is increasing on $(-\infty, 2)$
decreasing on $(2, 3]$
has local maximum at $x = 2$.

(c) (2 marks) Determine intervals where $f(x)$ is concave upwards or downwards, and the x-coordinates of inflection points (if any). You may use, without verifying it, the formula $f''(x) = (3x-12)(3-x)^{-3/2}/4$.

$f''(x)$ has same sign as $x-4$ so $f''(x) < 0$ for all $x \leq 3$
$\Rightarrow$ f is concave down on its entire domain, no inflection points

Question 4 continued on the next page...

Continued on page 9

Question 4 continued

(d) (2 marks) There is a point at which the tangent line to the curve $y = f(x)$ is vertical. Find this point.

$$\lim_{x \to 3^-} f'(x) = -\infty$$

Answer $x = 3$

(e) (2 marks) The graph of $y = f(x)$ has no asymptotes. However, there is a real number a for which $\lim_{x \to -\infty} \frac{f(x)}{|x|^a} = -1$. Find the value of a.

for $|x|$ large, $3 - x \approx |x|$,
so $f(x) \approx x \cdot |x|^{1/2} \approx x^{3/2}$

Answer $a = 3/2$

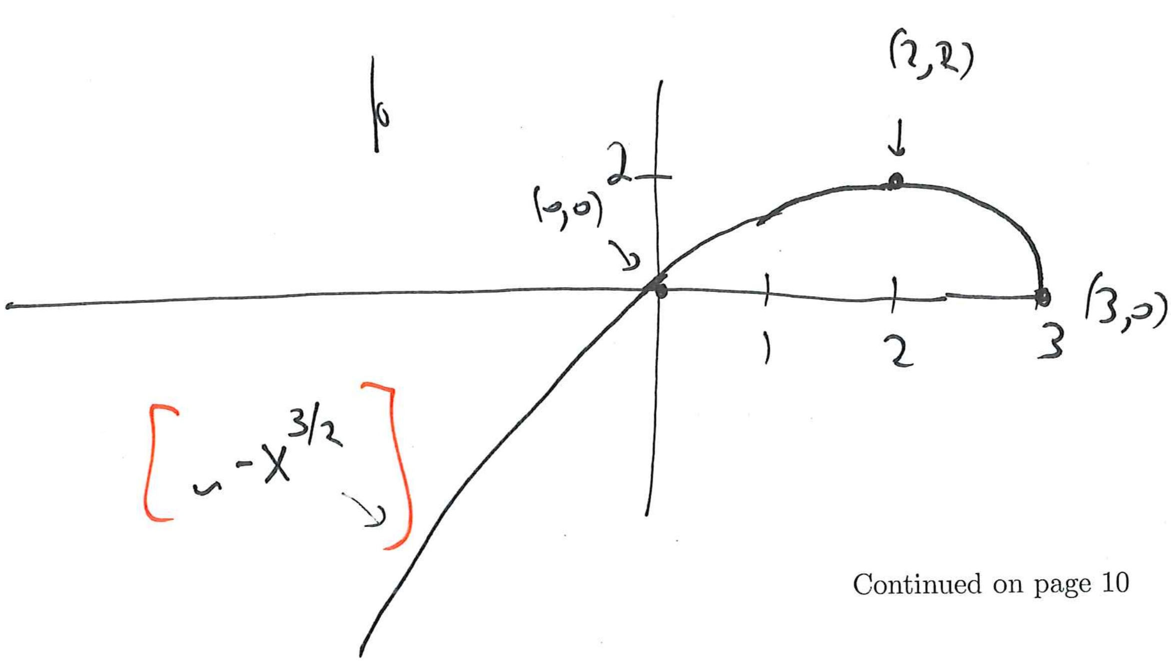
check: $\lim_{x \to -\infty} \frac{f(x)}{|x|^{3/2}} = \lim_{x \to -\infty} \frac{x\sqrt{3-x}}{|x|^{3/2}} = \lim_{x \to -\infty} \frac{x}{|x|} \cdot \frac{\sqrt{3-x}}{\sqrt{-x}} = \lim_{x \to -\infty} (-1) \cdot \sqrt{1 - \frac{3}{x}}$

$= -1$

$|x|^{3/2} = (-x)^{3/2} = (-x)(-x)^{1/2}$ / $\sqrt{-x}$

(f) (4 marks) Sketch the graph of $y = f(x)$, showing the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above and also all x-intercepts.

$f(3) = 0$, $f(2) = 2\sqrt{3-2} = 2$, $f(x) = 0$ for $x = 3, 0$



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[14] **4.** Let

$$f(x) = \begin{cases} \frac{4}{\pi}\tan^{-1} x, & \text{if } x \geq 1, \\ 2 - x^4, & \text{if } x < 1. \end{cases}$$

[*Note:* Another notation for $\tan^{-1}$ is arctan.]

(a) (3 marks) Show that $f(x)$ is continuous at $x = 1$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2-x^4) = 2-1^4 = 1 = f(1)$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{4}{\pi}\arctan x = \frac{4}{\pi}\arctan(1) = f(1) = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1$ } so f is cts at x=1

arctan is cts

(b) (1 mark) Determine the equations of any asymptotes (horizontal, vertical or slant).

f cts everywhere so no vertical asymptotes

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4}{\pi}\arctan x = \frac{4}{\pi}\cdot\frac{\pi}{2} = 2 \Rightarrow$ have horizontal asymptote $y=2$ as $x\to\infty$

As $x\to-\infty$ $f(x) \sim -x^4$ no asymptote

(c) (4 marks) Determine all critical numbers, open intervals where f is increasing or decreasing, and the x-coordinates of all local maxima or local minima (if any).

$$f'(x) = \begin{cases} -4x^3 & x<1 \\ \frac{4}{\pi(1+x^2)} & x>1 \end{cases}$$

at $x=1$, $\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \frac{4}{\pi(1+1^2)} = \frac{2}{\pi}$

$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = -4\cdot 1^3 = -4$

$f'(1)$ DNE.

$f'(0)=0$ so critical pt at $x=0$ ($-4x^3 \neq 0$ else)

singular pt at $x=1$

$\frac{4}{\pi}\frac{1}{1+x^2} > 0$ for all x

On $(-\infty, 0)$, $f'>0$ $(-x^3>0)$, on $(0,1)$ $f'<0$

On $(1,\infty)$, $f'>0$

Question 4 continues on the next page...

Continued on page 9

So f increasing on $(-\infty,0)$, has local max at $x=0$, decreasing on $(0,1)$

has local min at $x=1$ increasing on $(1,\infty)$

Question 4 continued

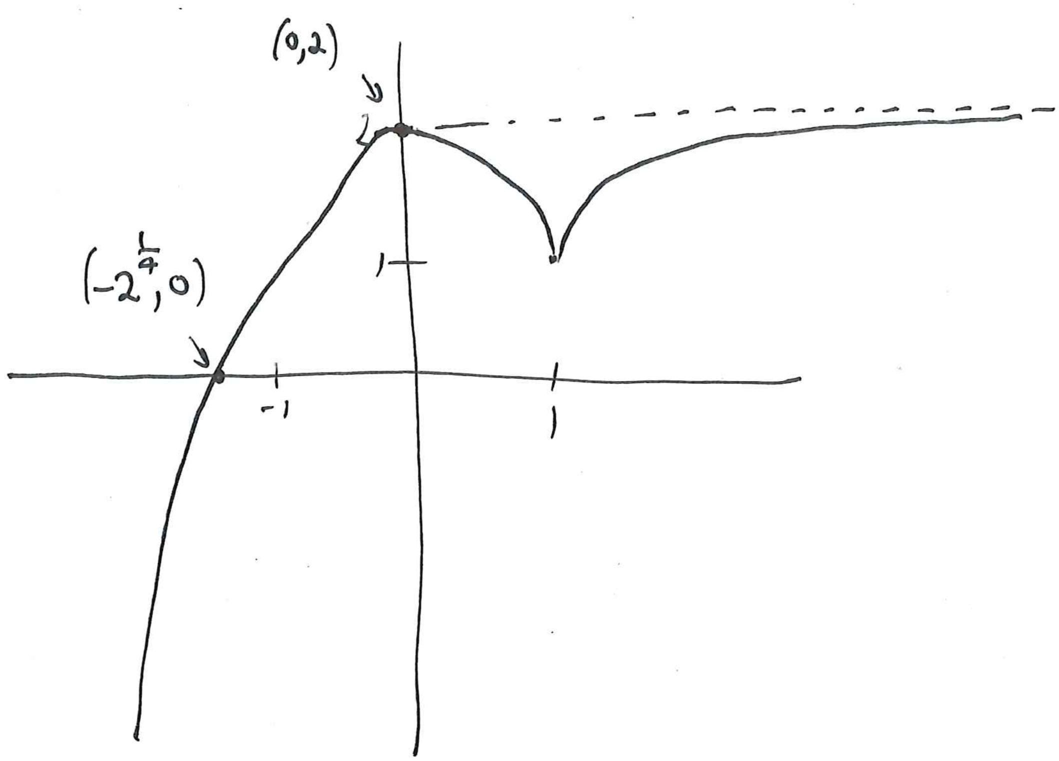
(d) (2 marks) Determine open intervals where the graph of f is concave upwards or concave downwards, and the x-coordinates of all inflection points (if any).

We have $f''(x) = \begin{cases} -\frac{4}{\pi}\frac{1}{(1+x^2)^2}\cdot 2x = -\frac{8x}{\pi(1+x^2)^2} & x>1 \\ -12x^2 & x<1 \end{cases}$

Both expressions are negative ($8x>0$ if $x>1$) except $f''(0)=0$.

f is therefore concave down on $(-\infty,1)$ and on $(1,\infty)$, and has no inflection points

(e) (4 marks) Sketch the curve $y = f(x)$, showing all the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above (if any).



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