Reminders: (1) Pre-class reading
(2) Post-class exercises

3. The extended sense; limits at infinity (16/9/2021)

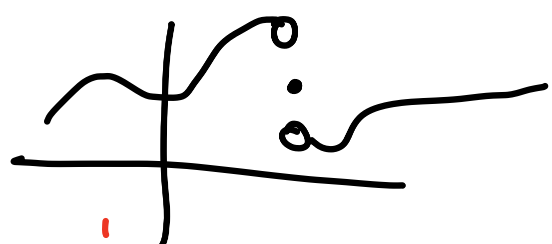
Goals.

(1) The squeeze theorem (carryover from last time)
(2) Infinite limits
   (a) Identify blowups
   (b) Examining signs to tell $\pm\infty$.
(3) Limits at infinity
   (a) "Gut feeling" approach to limits
   (b) Formal evaluation of limits at $\pm\infty$

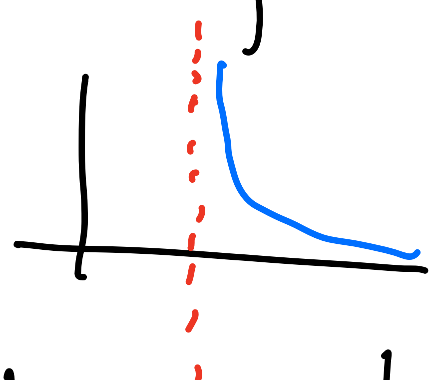
Last Time.

① Limits, geometrically: can see on the graph that $\lim_{x\to a} f(x)$ DNE:

(1) jump



(2) blowup:



(3) wild oscillation:

(*) Calculating limits using "limit laws"
= "arithmetic of limits".

(*) algebraic trick: $\sqrt{a} - \sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$

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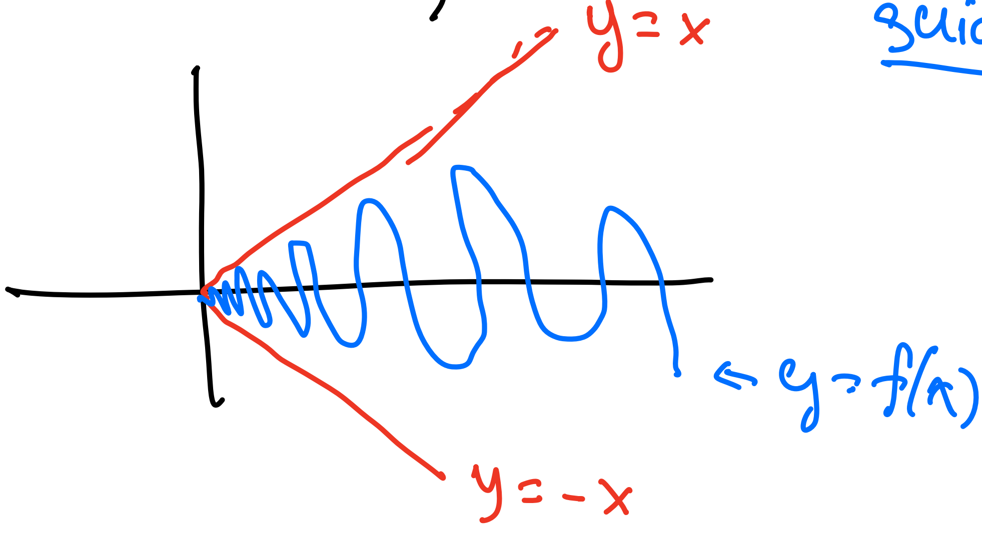
① The squeeze thm.

Warning: difficulty & discomfort with inequalities

Idea: sometimes hard to control f directly. But f stays between guidelines



In this example, if x is small,

$-x \le f(x) \le x$

so $f(x)$ is small too.

$\Rightarrow \lim_{x \to 0^+} f(x) = 0$

<u>Formally</u>: Given f, look for "guidelines": functions g, h s.t. $g \le f \le h$

and $\lim_{x \to a} g(x)$, $\lim_{x \to a} h(x)$ are easier to calculate & <u>equal</u>.

Then The Squeeze Thm says that $\lim_{x \to a} f(x)$ equals this number too.

Examples: (1) Find $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$

(2) (Final, 2014): Suppose

$$8x \le f(x) \le x^2+16 \text{ for all } x \ge 0.$$

Find $\lim_{x\to 4} f(x)$.

Solutions:

(1) [$\sin\left(\frac{\pi}{x}\right)$ does not make sense if $x=0$; highly oscillatory near 0].

[Can't say: $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = \left(\lim_{x\to 0} x^2\right)\left(\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)\right) = 0.(\ ) = 0$]

this limit DNE

[But $\lim_{x\to 0} x^2 = 0$ still useful:

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$]

For all $x \neq 0$, $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$.

Therefore

$$-x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2$$

(multiplied inequality by positive quantity x^2)

Now $\lim_{x \to 0} x^2 = 0^2 = 0$, $\lim_{x \to 0} (-x^2) = -0^2 = 0$

By the squeeze thm,

$$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$$

as well.

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(2) $\lim_{x \to 4} 8x = 8 \cdot 4 = 32$

$\lim_{x \to 4} x^2 + 16 = 4^2 + 16 = 32$

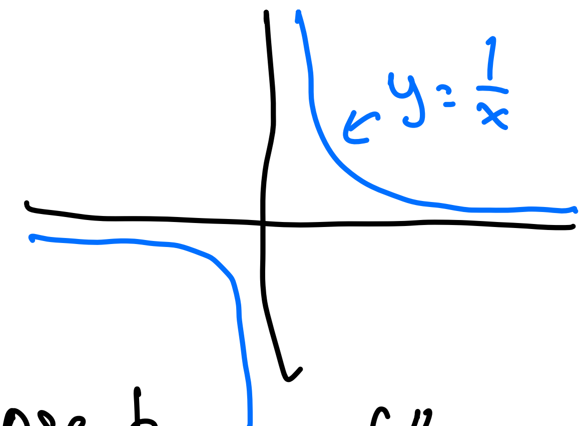
Also, near 4, $8x \le f(x) \le x^2 + 16$

By the squeeze thm $\lim_{x \to 4} f(x) = 32$ too.

② Blowups

Sometimes f "blows up" at a: as x gets closer to a, $f(x)$ escapes to ∞, $-\infty$.

Example:



If x is close to 0 ("very small")
Then $\frac{1}{x}$ is very large
("close to ∞" or "close to $-\infty$")

If $a \neq 0$, $\frac{1}{x}$ makes sense at a,
and $\lim_{x \to a} \frac{1}{x} = \frac{1}{a}$.

But $\lim_{x \to 0^+} \frac{1}{x}$ DNE. We write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$ (= "limit DNE, but fcn blows up toward $+\infty$")

similarly $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

Example: $f(x) = \frac{e^x}{x^2 - x}$.

At which point might f blow up?

– if $x^2 - x = 0$, i.e. if $x = 0$ or $x = 1$

What happens?

review: If function blows up (denominator $\to 0$, numerator $\to \neq 0$) need to check sign of f to see if we're going to ∞ or $-\infty$.

If $x > 0$, x close to 0,
e^x is close to $e^0 = 1$.

$x^2 - x = x(x-1)$
is close to $-x$ since $x-1$
is close to -1.

[or: $f(x) = \dfrac{e^x / 1-x}{x}$, $\dfrac{e^x}{1-x} \xrightarrow[x \to 0]{} -1$]

So if $x > 0$, $\frac{1}{x} > 0$ large,
$\dfrac{e^x}{x^2 - x}$ is negative & large

So $\displaystyle\lim_{x \to 0^+} \frac{e^x}{x^2 - x} = -\infty$

Similarly if $x < 0$, $\dfrac{e^x}{x(1-x)} > 0$

\+ close to 0

so $\displaystyle\lim_{x \to 0^-} \frac{e^x}{x(1-x)} = +\infty$.

Similarly near $x=1$:

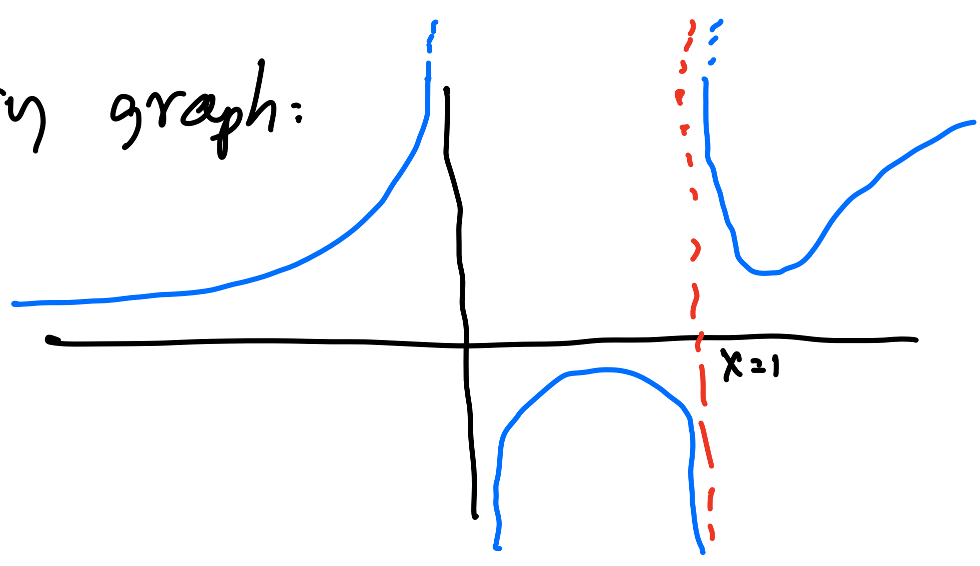
if x is close to 1, $\dfrac{e^x}{x} \xrightarrow[x\to 1]{} \dfrac{e^1}{1} = e > 0$

so near $x=1$, $\dfrac{e^x/x}{x-1} \begin{cases} <0 & \text{if } x<1 \\ >0 & \text{if } x>1 \end{cases}$

so $\dfrac{e^x}{x(x-1)}$ blows up at $x=1$

with
$$\lim_{x\to 1^+} \frac{e^x}{x^2-x} = \infty$$
$$\lim_{x\to 1^-} \frac{e^x}{x^2-x} = -\infty$$

Try graph:



Math 100 – WORKSHEET 3
INFINITE LIMITS AND LIMITS AT INFINITY

1. Infinite limits

(1)
(a) (Final, 2014) Evaluate $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

If x is close to -3, $x+2$ is close to -1, $x+3$ is close to 0, so expression blows up at -3. If $x>-3$, $x+3>0$, so $\frac{x+2}{x+3}<0$, so $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$

(b) Let $f(x) = \frac{x-3}{x^2+x-12}$. What is $\lim_{x\to \cancel{2}\,-4} f(x)$? What about $\lim_{x\to -\cancel{2}\,4^+} f(x)$, $\lim_{x\to \cancel{2}\,-4^-} f(x)$?

$f(x) = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4}$, so f blows up at -4. If $x>-4$, $\frac{1}{x+4}>0$; if $x<-4$, $\frac{1}{x+4}<0$

so $\lim_{x\to -4^+} \frac{1}{x+4} = \infty$, $\lim_{x\to -4^-} \frac{1}{x+4} = -\infty$,

Date: 16/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$\lim_{x\to -4} \frac{1}{x+4}$ DNE, even in the extended sense.

(2) Evaluate

(a) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

This blows up at $x=1$, and $\frac{1}{(x-1)^2}>0$ near $x=1$, so

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$$

(b) $\lim_{x\to 2} \frac{\sin x}{|x-2|}$

(c) $\lim_{x\to \frac{\pi}{2}^+} \tan x$, $\lim_{x\to \frac{\pi}{2}^-} \tan x$.

③ Limits at ∞

Sometimes, clear that

$$f(x) \xrightarrow[x\to\infty]{} \infty$$

Examples: $f(x) = x$, $f(x) = x^7$

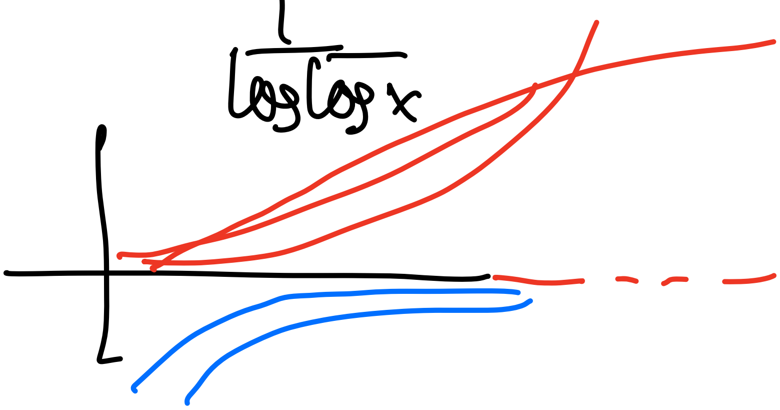
$f(x) = e^x$

also $\log x$, $\sqrt{x}$

Sometimes, clear that $f(x) \xrightarrow[x\to\infty]{} 0$

Examples: $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{e^x + x^7}$, $\frac{1}{\log x}$,

$\frac{1}{\log \log x}$



$-\frac{1}{x^{3/2}}$

Interesting: Have a race:
different parts of f
"pull" in different directions

Example: $\frac{e^x}{x}$, $x^7 - x^3$, $\frac{x}{x^2+1}$,
$\frac{5x+1}{3x+2}$

Two goals: ① "gut method":
look & see who wins.
② formal calculation

Know: exponentials beat
$e^x, e^{5x}, e^{x/30}$
power laws $x^{70}, x^{1/2}$, which
+ logarithms

Eg. $\frac{e^x}{x} \xrightarrow[x\to\infty]{} \infty$, also $\frac{e^x}{x^{1000}}$.

(reason: exponential asymptotically larger that power law)

$x^7 - x^3$? clearly x^7 is much larger than x^3 (if x is large), so this will behave like x^7.

$$\frac{x}{x^2+1} \sim_\infty \frac{x}{x^2} = \frac{1}{x} \xrightarrow[x\to\infty]{} 0$$

(not acceptable as solution)

$$\frac{5x+1}{3x-2} \sim_\infty \frac{5x}{3x} = \frac{5}{3}$$

(2) Acceptable justification: "extracting asymptotics":

Example: look at $x^7 - x^3$.

Our gut says x^7 is dominant, so we take common factor of x^7:

$$x^7 - x^3 = x^7\left(1 - \frac{1}{x^4}\right)$$

Now $x^7 \xrightarrow[x\to\infty]{} \infty$, $\quad 1 - \frac{1}{x^4} \xrightarrow[x\to\infty]{} 1 - 0 = 1$

So $x^7 - x^3 \xrightarrow[x\to+\infty]{} \infty$

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similarly as $x \to -\infty$, $1 - \frac{1}{x^4} \xrightarrow[x\to-\infty]{} 1 - 0 = 1$

But x^7 is huge & negative

So $x^7 - x^3 = x^7\left(1 - \frac{1}{x^4}\right) \xrightarrow[x\to-\infty]{} -\infty$

Another example:

$$\lim_{x\to\infty} \frac{5x+1}{3x-2} = \lim_{x\to\infty} \frac{\cancel{x}\left(5+\frac{1}{x}\right)}{\cancel{x}\left(3-\frac{2}{x}\right)} = \lim_{x\to\infty} \frac{5+\frac{1}{x}}{3-\frac{2}{x}}$$

$$= \frac{5+0}{3-0} = \frac{5}{3}.$$

2. Limits at infinity

(1) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3} =$

["sat": $x^2+1 \sim_\infty x^2$, $x-3 \sim_\infty x$]

$$\frac{x^2+1}{x-3} = \frac{x^2(1+\frac{1}{x^2})}{x(1-3/x)} = x \cdot \frac{1+\frac{1}{x^2}}{1-3/x} \xrightarrow[x\to\infty]{} \infty$$

Because $x \to \infty$, $\frac{1+\frac{1}{x^2}}{1-3/x} \xrightarrow[x\to\infty]{} 1 > 0$

(b) (Final, 2015) $\lim_{x\to\infty} \frac{x+1}{x^2+2x-8} =$

$$\frac{x+1}{x^2+2x-8} = \frac{x(1+\frac{1}{x})}{x^2(1+\frac{2}{x}-\frac{8}{x^2})} = \left(\frac{1}{x}\right) \cdot \frac{1+\frac{1}{x}}{1+2/x-8/x^2} \xrightarrow[x\to\infty]{} 0 \cdot \frac{1+0}{1+0-0} = 0.$$

$\sqrt{y^2} = |y|$

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} =$

[$4x^2+x \sim_{-\infty} 4x^2$, so $\sqrt{4x^2+x} \sim_{-\infty} \sqrt{4x^2} = |2x| = -2x$]

so $\frac{3x}{\sqrt{4x^2+x}-2x} \sim_{-\infty} \frac{3x}{-2x-2x} = -\frac{3}{4}$]

everything on scale x.

$$\frac{3x}{\sqrt{4x^2+x}-2x} = \frac{3x}{\sqrt{x^2}\sqrt{4+\frac{1}{x}}-2x} = \frac{3x}{-x\sqrt{4+\frac{1}{x}}-2x} = \frac{-3}{\sqrt{4+\frac{1}{x}}+2}$$

if $x<0$, $\sqrt{x^2} = -x$

$x \to -\infty$

$$\frac{-3}{\sqrt{4+0}+2} = -\frac{3}{4}.$$

(d) $\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} =$

[$x^4+\sin x \sim x^4$

$x^2-\cos x \sim x^2$ $\quad \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} \sim \frac{\sqrt{x^4}}{x^2} = 1$]

$$\frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \frac{\cancel{\sqrt{x^4}}\sqrt{1+\frac{\sin x}{x^4}}}{\cancel{x^2}\cdot\left(1-\frac{\cos x}{x^2}\right)} = \frac{\sqrt{1+\frac{\sin x}{x^4}}}{1-\frac{\cos x}{x^2}}$$

(use squeeze thm to show $\lim_{x\to\infty} \frac{\sin x}{x^4} = 0$

$\lim_{x\to\infty} \frac{\cos x}{x^2} = 0$, so overall get $\frac{\sqrt{1+0}}{1-0} = 1$.

(e) $\lim_{x\to-\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-1}\right) =$

$\sqrt{x^2+2x} \sim_{-\infty} \sqrt{x^2} \sim_{-\infty} -x$

$\sqrt{x^2-1} \sim_{-\infty} \sqrt{x^2} \sim_{-\infty} -x$

$$\left\{\begin{array}{l} (-x)-(-x+1) = -1 \\ (-x)-(-x+\sqrt{x}) = -\sqrt{x} \\ (-x)-(-x-\sqrt{x}) = +\sqrt{x} \end{array}\right.$$

need to compute cancellation!

$$\sqrt{x^2+2x}-\sqrt{x^2-1} = \frac{(x^2+2x)-(x^2-1)}{\sqrt{x^2+2x}+\sqrt{x^2-1}}$$

$$= \frac{2x+1}{\sqrt{x^2+2x}+\sqrt{x^2-1}} \quad \div\, - \quad \xrightarrow[x\to\infty]{} \frac{2}{1+1} = 1$$