Math 100 – SOLUTIONS TO WORKSHEET 3
INFINITE LIMITS AND LIMITS AT INFINITY

1. Infinite limits

(1)

(a) (Final, 2014) Evaluate $\lim_{x\to-3^+}\frac{x+2}{x+3}$.
Solution: The denominator vanishes at -3 while the numerator does not, so the function blows up there. When $x>-3$, we have $x+3>0$. Also, when x is close to -3, $x+2$ is close to -1. We conclude that $\lim_{x\to-3^+}\frac{x+2}{x+3}=-\infty$.

(b) Let $f(x)=\frac{x-3}{x^2+x-12}$. What is $\lim_{x\to4}f(x)$? What about $\lim_{x\to-4^+}f(x)$, $\lim_{x\to-4^-}f(x)$?
Solution: The limits do not exist: if x is very close to -4 then $x+4$ is very small and $\frac{x-3}{x^2+x-12}=\frac{x-3}{(x-3)(x+4)}=\frac{1}{x+4}$ is very large. That said, when $x>-4$ we have $\frac{1}{x+4}>0$ and when $x<-4$ we have $\frac{1}{x+4}<0$ so (in the extended sense)

$$\lim_{x\to-4^+}\frac{x-3}{x^2+x-12}=+\infty$$
$$\lim_{x\to-4^-}\frac{x-3}{x^2+x-12}=-\infty\,.$$

(2) Evaluate

(a) $\lim_{x\to1}\frac{1}{(x-1)^2}$
Solution: The function blows up at both sides, and remains positive on both sides. Therefore

$$\lim_{x\to1}\frac{1}{(x-1)^2}=\infty\,.$$

(b) $\lim_{x\to4}\frac{\sin x}{|x-4|}$
Solution: $|x-4|\to0$ as $x\to4$ while $\sin x\xrightarrow[x\to4]{}\sin4\neq0$, so the function blows up there. Since $|x-2|$ is positive and $\sin4$ is negative ($\pi<4<2\pi$) we have

$$\lim_{x\to4}\frac{\sin x}{|x-4|}=-\infty\,.$$

(c) $\lim_{x\to\frac{\pi}{2}^+}\tan x$, $\lim_{x\to\frac{\pi}{2}^-}\tan x$.
Solution: We have $\tan x=\frac{\sin x}{\cos x}$. Now for x close to $\frac{\pi}{2}$, $\sin x$ is close to $\sin\frac{\pi}{2}=1$, so $\sin x$ is positive. On the other hand $\lim_{x\to\frac{\pi}{2}}\cos x=\cos\frac{\pi}{2}=0$ so $\tan x$ blows up there. Since $\cos x$ is decreasing on $[0,\pi]$ it is positive if $x<\frac{\pi}{2}$ and negative if $x>\frac{\pi}{2}$, so:

$$\lim_{x\to\frac{\pi}{2}^+}\tan x=-\infty$$
$$\lim_{x\to\frac{\pi}{2}^-}\tan x=+\infty$$

2. Limits at infinity

(1) Evaluate the following limits:

(a) $\lim_{x\to\infty}\frac{x^2+1}{x-3}=$
Solution: $\lim_{x\to\infty}\frac{x^2+1}{x-3}=\lim_{x\to\infty}\frac{x^2}{x}\cdot\frac{1+\frac{1}{x^2}}{1-\frac{3}{x}}=\lim_{x\to\infty}x\cdot\frac{1+\frac{1}{x^2}}{1-\frac{3}{x}}=\infty$.

(b) (Final, 2015) $\lim_{x\to\infty}\frac{x+1}{x^2+2x-8}=$
Solution: $\lim_{x\to\infty}\frac{x+1}{x^2+2x-8}=\lim_{x\to\infty}\frac{x}{x^2}\cdot\frac{1+\frac{1}{x}}{1+\frac{2}{x}-\frac{8}{x^2}}=\lim_{x\to\infty}\frac{1}{x}\cdot\frac{1+\frac{1}{x}}{1+\frac{2}{x}-\frac{8}{x^2}}=0$.

Date: 16/9/2021, Worksheet by Lior Silberman.

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} =$

Solution: We have

$$\begin{aligned}\frac{3x}{\sqrt{4x^2+x}-2x} &= \frac{3x}{\sqrt{x^2\left(4+\frac{x}{x^2}\right)}-2x} = \frac{3x}{\sqrt{x^2}\sqrt{4+\frac{1}{x}}-2x} \\ &= \frac{3x}{|x|\sqrt{4+\frac{1}{x}}-2x} = \frac{3x}{(-x)\sqrt{4+\frac{1}{x}}-2x} \\ &= \frac{3}{-\sqrt{4+\frac{1}{x}}-2} \xrightarrow[x\to-\infty]{} \frac{3}{-\sqrt{4+0}-2} = \boxed{-\frac{3}{4}}.\end{aligned}$$

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(d) $\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} =$

Solution: $\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \lim_{x\to\infty} \frac{x^2\sqrt{1+\frac{\sin x}{x^4}}}{x^2\left(1-\frac{\cos x}{x^2}\right)} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{\sin x}{x^4}}}{1-\frac{\cos x}{x^2}}$. Now for all x we have $-\frac{1}{x^4} \le \frac{\sin x}{x^4} \le \frac{1}{x^4}$ and $-\frac{1}{x^2} \le \frac{\cos x}{x^2} \le \frac{1}{x^2}$. Since $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^4} = 0$ by the squeeze theorem we have $\lim_{x\to\infty} \frac{\sin x}{x^4} = \lim_{x\to\infty} \frac{\cos x}{x^2} = 0$. Thus

$$\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \frac{\sqrt{1+0}}{1-0} = \boxed{1}.$$

(e) $\lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-1}\right) =$

Solution: We have

$$\begin{aligned}\sqrt{x^2+2x}-\sqrt{x^2-1} &= \sqrt{x^2+2x}-\sqrt{x^2-1}\cdot\frac{\sqrt{x^2+2x}+\sqrt{x^2-1}}{\sqrt{x^2+2x}+\sqrt{x^2-1}} = \frac{\left(x^2+2x\right)-\left(x^2-1\right)}{\sqrt{x^2+2x}+\sqrt{x^2-1}} \\ &= \frac{2x+1}{|x|\sqrt{1+\frac{2}{x}}+|x|\sqrt{1-\frac{1}{x^2}}} = \frac{x\left(2+\frac{1}{x}\right)}{-x\sqrt{1+\frac{2}{x}}+(-x)\sqrt{1-\frac{1}{x^2}}} \\ &= -\frac{2+\frac{1}{x}}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x^2}}} \xrightarrow[x\to-\infty]{} -\frac{2}{1+1} = \boxed{-1}.\end{aligned}$$