Math 100 – SOLUTIONS TO WORKSHEET 5
THE IVT

1. The Intermediate Value Theorem

(1) Show that $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.

Solution: f is continuous on $[0, 1]$ (given by formula there). We have $f(0) = 1$, $f(1) = -2$. By the intermediate value theorem there is $x_0 \in (0, 1)$ such that $f(x_0) = 0 \in (-2, 1)$.

(2) (Final 2011) Let $y = f(x)$ be continuous with domain $[0, 1]$ and range in $[3, 5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

Solution: Consider the diffference $g(x) = f(x) - (2x + 3)$. By arithmetic of limits this is a continuous function. We have $g(0) = f(0) - 3 \ge 3 - 3 = 0$ (since $f(0) \ge 3$). We have $g(1) = f(1) - 5 \le 5 - 5 = 0$. By the IVT $g(x)$ takes every value between $g(0)$ and $g(1)$, so there is x_0 such that $g(x_0) = 0$ and then $f(x_0) - (2x_0 + 3) = 0$ so $f(x_0) = 2x_0 + 3$ so the graphs intersect at the point $(x_0, 2x_0 + 3)$.

(3) $\sin x = x + 1$ has a solution.

Solution: Let $f(x) = x + 1 - \sin x$, so we want x such that $f(x) = 0$. The function f is continuous. Note that $f(100) = 101 - \sin 100 \ge 100$ while $f(-100) = -100 + 1 - \sin 100 \le -98$. By the IVT there is $x_0 \in (-100, 100)$ where $f(x_0) = 0$, that is $x_0 + 1 - \sin x_0 = 0$ so $x_0 + 1 = \sin x_0$.

(4) (Final 2015) Show that the equation $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions.

Solution: We have $f(0) = 0 - 3 + 0 + 1 = -2$. On the other hand if x has large magnitude then $f(x)$ is positive:

$$f(10) = 200 - 3 + \sin 10 + \cos 10 \ge 200 - 5 = 195\,.$$
$$f(-10) = 200 - 3 - \sin 10 + \cos 10 \ge 200 - 5 = 195\,.$$

Thus $f(-10), f(10)$ are positive, $f(0)$ is negative. Since f is continuous everywhere (given by formula), the IVT shows that its graph crosses the x axis once in $(-10, 0)$ and once in $(0, 10)$.

(5) (Final 2018) Let g be a continuous function such that

$$\frac{x}{2} \le g(x) \le \frac{x}{2} + 1$$

for each positive real number x. Let $f(x) = g(x) + \sin x$. Show that there are infinitely many real numbers c such that $f(c) = \frac{c+1}{2}$.

Solution: Let $h(x) = f(x) - \frac{x+1}{2}$, in terms of which we are looking for points c such that $h(c) = 0$. Then

$$h(x) = (g(x) + \sin x) - \frac{x+1}{2} = \left(g(x) - \frac{x+1}{2}\right) + \sin x\,.$$

We now note that for all x we have

$$\frac{x}{2} \le g(x) \le \frac{x}{2} + 1$$

and therefore

$$-\frac{1}{2} \le g(x) - \frac{x}{2} - \frac{1}{2} \le \frac{1}{2}\,.$$

It follows that whenever $\sin x = 1$ we have

$$h(x) = \left(g(x) - \frac{x+1}{2}\right) + 1 \ge -\frac{1}{2} + 1 \ge \frac{1}{2} > 0$$

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and whenver $\sin x = -1$ we have

$$h(x) = \left(g(x) - \frac{x+1}{2}\right) - 1 \le \frac{1}{2} - 1 \le -\frac{1}{2} < 0\,.$$

Accordingly for $k \in \mathbb{Z}$ let $a_k = 2\pi k + \frac{\pi}{2}$ and let $b_k = 2\pi k + \frac{3\pi}{2}$. Then $\sin a_k = 1$, $\sin b_k = -1$ and therefore

$$h(a_k) > 0, \qquad h(b_k) < 0\,.$$

Now h is a continuous function ($g(x)$ is assumed continuous and h is the sum of g and a function defined by formula and continuous everywhere). It then follows from the IVT that for each k there is $c_k \in (a_k, b_k)$ such that $h(c_k) = 0$ or equivalently $f(c_k) = \frac{c_k+1}{2}$. Finally the values c_k are all different (they each belong to a different cycle of the sine function) so there are indeed infinitely many of them.

2. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

(6) Find $f'(a)$ if
  (a) $f(x) = x^2$, $a = 3$.
  Solution: $\lim_{h\to 0} \frac{(3+h)^2-(3)^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h) = 6$.
  (b) $f(x) = \frac{1}{x}$, any a.
  Solution: $\lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right) = \lim_{h\to 0} \frac{-h}{h\cdot a(a+h)} = -\lim_{h\to 0} \frac{1}{a(a+h)} = -\frac{1}{a^2}$.
  (c) $f(x) = x^3 - 2x$, any a. (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).
  Solution: We have

$$\begin{aligned}\frac{(a+h)^3 - 2(a+h) - a^3 + 2a}{h} &= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h - a^3 + 2a}{h} \\ &= \frac{3a^2h + 3ah^2 + h^3 - 2h}{h} \\ &= 3a^2 - 2 + 3ah + h^2 \xrightarrow[h\to 0]{} 3a^2 - 2\,.\end{aligned}$$

(7) Express the limit as a derivative: $\lim_{h\to 0} \frac{\cos(5+h)-\cos 5}{h}$.
  Solution: This is the derivative of $f(x) = \cos x$ at the point $a = 5$.