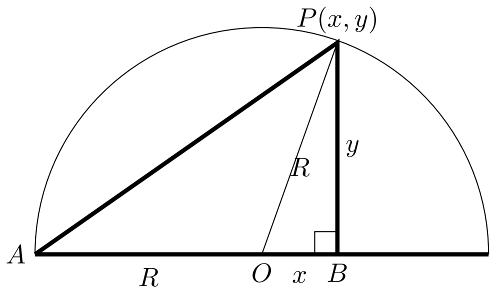
Math 100 – SOLUTIONS TO WORKSHEET 17
OPTIMIZATION

(1) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1-x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of this triangle?

Solution: (0) Picture



(1) Put the coordinate system where the centre of the circle is at $(0, 0)$ and the diameter is on the x-axis. Let B be at $(x, 0)$, P at (x, y).
(2) Since P is on the circle we have $y = \sqrt{1-x^2}$. The area of the triangle is then $A = \frac{1}{2}(\text{base}) \times (\text{height}) = \frac{1}{2}(1+x)\sqrt{1-x^2}$ since the base of the triangle has length $1+x$.
(4) The function $A(x)$ is continuous on $[-1, 1]$ so we can find its minimum by differentiation. By the product rule and chain rule,

$$\begin{aligned} A'(x) &= \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(1+x)\frac{-2x}{2\sqrt{1-x^2}} \\ &= \frac{\left(\sqrt{1-x^2}\right)^2}{2\sqrt{1-x^2}} - \frac{x(1+x)}{2\sqrt{1-x^2}} = \frac{1-x^2-x-x^2}{2\sqrt{1-x^2}} \\ &= \frac{1-x-2x^2}{2\sqrt{1-x^2}} . \end{aligned}$$

This is defined on $(-1, 1)$ and the critical points satisfy $2x^2 + x - 1 = 0$ so they are $x = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4} = -1, \frac{1}{2}$. The only critical point in the interior is then $x = \frac{1}{2}$. The area vanishes at the endpoints (the triangle becomes degenerate) and

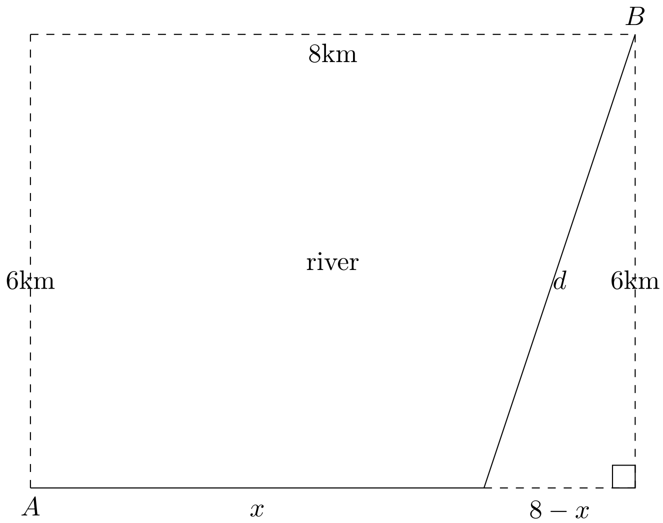
$$A\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{1 - \frac{1}{2^2}} = \frac{3\sqrt{3}}{8} .$$

It follows that the largest possible area is $\frac{3\sqrt{3}}{8}$.

Date: 9/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?

Solution: (0) Picture



(1) Build a road of length x from A along the bank, then build a bridge of length d toward B.
(2) By Pythagoras, $d = \sqrt{6^2 + (8-x)^2}$.
(3) The total cost is

$$C(x) = 20x + 40\sqrt{6^2 + (8-x)^2} = 20x + 40\sqrt{6^2 + (x-8)^2}\,.$$

(4) The function $C(x)$ is defined everywhere ($6^2 + (8-x)^2 \geq 6^2 > 0$) and continuous there. We have

$$C'(x) = 20 + 40\frac{2(x-8)}{2\sqrt{6^2 + (x-8)^2}}\,.$$

This exists everywhere (the denominator is everywhere positive by the same calculation). It's enough to consider $0 \leq x \leq 8$ (no point in starting the bridge west of A or east of B). Looking for critical points we solve $C'(x) = 0$ that is:

$$\begin{aligned}
20 + 40\frac{x-8}{\sqrt{36 + (x-8)^2}} &= 0\\
20 &= 40\frac{8-x}{\sqrt{36 + (8-x)^2}}\\
\sqrt{36 + (8-x)^2} &= 2(8-x)\\
36 + (8-x)^2 &= 4(8-x)^2\\
36 &= 3(8-x)^2\\
(8-x) &= \sqrt{\frac{36}{3}} = \sqrt{12} = 2\sqrt{3}
\end{aligned}$$

(only the positive root since $0 \leq x \leq 8$ forces $8 - x \geq 0$) so

$$x = 8 - 2\sqrt{3}\,.$$

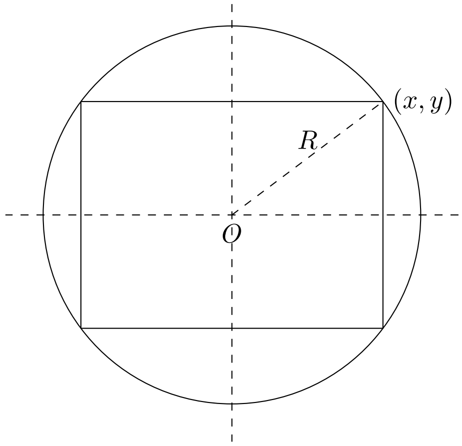
We then have $C(0) = 40\sqrt{6^2 + 8^2} = 40\sqrt{100} = 400$, $C(8) = 20 \cdot 8 + 40\sqrt{6^2} = 160 + 240 = 400$ and

$$\begin{aligned}
C(8 - 2\sqrt{3}) &= 20\left(8 - 2\sqrt{3}\right) + 40\sqrt{6^2 + (2\sqrt{3})^2} = 160 - 40\sqrt{3} + 40\sqrt{36 + 12}\\
&= 160 - 40\sqrt{3} + 40\sqrt{48} = 160 - 40\sqrt{3} + 40\sqrt{16 \cdot 3}\\
&= 160 - 40\sqrt{3} + 40 \cdot 4\sqrt{3} = 160 + 120\sqrt{3}\,.
\end{aligned}$$

Now $\sqrt{3} < \sqrt{4} = 2$ so $C(8 - 2\sqrt{3}) = 160 + 120\sqrt{3} < 160 + 120 \cdot 2 = 400 = C(0) = C(8)$ and we conclude that $C(8 - 2\sqrt{3})$ is the minimum.
(5) The cheapest way to construct a bridge is construct a road of length $\left(8 - 2\sqrt{3}\right)$ km along the bank from A toward B, and then bridge from the end of the road to B.

(3) (Final 2019) Among all rectangles inscribed in a given circle, which one has the largest perimeter? Prove your answer.

Solution: (0) Picture



(1) We rotate the rectangle so that it's aligned with the axes; suppose one corner is at (x, y). Call the radius of the circle R and the perimeter of the rectangle P.
(2) We have $x^2 + y^2 = R^2$ so $y = \sqrt{R^2 - x^2}$.
(3) The total perimeter is

$$P(x) = 2x + 2y + 2x + 2y = 4(x + y) = 4(x + \sqrt{R^2 - x^2})$$

where $0 \le x \le R$.
(4) The function P is defined and continuous on $[0, R]$. We have

$$P'(x) = 4\left(1 - \frac{2x}{2\sqrt{R^2 - x^2}}\right)$$

This exists everywhere except at the endpoint $x = R$ where the denominator vanishes. There are critical points where $C'(x) = 0$ that is where

$$\begin{aligned} 4\left(1 - \frac{x}{\sqrt{R^2 - x^2}}\right) &= 0 \\ 1 &= \frac{x}{\sqrt{R^2 - x^2}} \\ \sqrt{R^2 - x^2} &= x \\ R^2 - x^2 &= x^2 \\ 2x^2 &= R^2 \\ x &= \frac{1}{\sqrt{2}} R\,. \end{aligned}$$

We have $P(\frac{1}{\sqrt{2}}R) = 4\left(\frac{1}{\sqrt{2}}R + \sqrt{R^2 - \frac{1}{2}R^2}\right) = 4\left(\frac{2}{\sqrt{2}}R\right) = 4\sqrt{2}R$ while at the endpoints we have $P(0) = 4\left(0 + \sqrt{R^2}\right) = 4R$ and $P(R) = 4\left(R + \sqrt{0}\right) = 4R$. It follows that the largest perimeter occurs when $x = \frac{1}{\sqrt{2}}R$.
(5) This rectangle also has $y = \sqrt{R^2 - x^2} = \frac{1}{\sqrt{2}}R$ so the rectangle with the largest perimeter is the square.