Lior Silberman's Math 223: Problem Set 1 (due 19/1/2022)

- Recommended practice problems are from the textbook by Friedberg, Insel and Spence.
- Problems M1,M2, etc are calculational practice: make sure you can solve them and problems like them automatically without much thinking. Problems like them may appear on exams.
- Numbered problems are for submissions; a problem with * or ** may be unusually difficult. For your convenience problems taken from that textbook above are so noted: (§1.3 E3,4) are problems 3,4 after section 1.3 of that book. RMK indicates a remark, not an exercise.
- Problems C1,C2, etc are for sumbission as an extra challenge.
- Lettered problems, as well as problems or subproblems labeled SUPP, are **supplementary** and not for submission; these generally cover additional ideas beyond the scope of the course.

Practice problems (recommended, but do not submit)

Section 1.2, problems 1-4, 8, 12-13, 17-19.
Section 1.3, problems 1-4, 8, 11, 16-17.

Calculational practice (also not for submission)

The following two problems review a skill from highschool: solving systems of linears equations in 1,2,3 unknowns (the second problem requires you to set the systems up, of course). You will use this skill repeatedly in the course. Eventually we will also address this topic systematically and in greater generality.

M1. Find all solutions in real numbers to the following equations: (a) $5x+7=13$
(b) $\begin{cases} 5x+2y=3 \\ 6x+4y=2 \end{cases}$ (c) $\begin{cases} 3x+2y=a \\ 6x+4y=a+1 \end{cases}$ (your answer may depend on the parameter a).

M2. In each of the following problems (1) Convert the equality of polynomials in x to a system of three linear equations in the unknown coefficients a,b,c; (2) either exhibit a solution (values for a,b,c) making the equality hold true (in which case no proof is needed) or prove that no such solution exists.
(a) $a\left(x^2+2x+1\right)+b\left(5x+3\right)+c\left(2\right)=7x^2-5x+3$;
(b) $a\left(x^2-2x+1\right)+b\left(x-1\right)+c\left(x^2+5x\right)=x^2+2x+3$;
(c) $a\left(x^2-2x+1\right)+b\left(x-1\right)+c\left(x^2-x\right)=x^2+2x+3$.

Subspaces

1. In each case decide if the set is a subspace of the given space.
(a) $U_1=\left\{\underline{x}\in\mathbb{R}^2 \mid 3x_1+2x_2=0\right\}\subset\mathbb{R}^2$, $U_2=\left\{\underline{x}\in\mathbb{R}^3 \mid x_1-7=x_3\right\}\subset\mathbb{R}^3$,
$U_3=\left\{\underline{x}\in\mathbb{R}^3 \mid x_1^2+x_2^2-x_3^2=0\right\}\subset\mathbb{R}^3$.
(b) $V_1=\left\{f\in\mathbb{R}^{\mathbb{R}} \mid \forall t: f(t)=2f(t+1)+1\right\}\subset\mathbb{R}^{\mathbb{R}}$,
$V_2=\left\{f\in\mathbb{R}^{\mathbb{R}} \mid f \text{ is differentiable and } e^x f'(x)+(\sin x)f(x)=0\right\}\subset\mathbb{R}^{\mathbb{R}}$.
RMK Example V_2 is the beginning of the theory of *linear differential equations*.

2. Fix a vector space V.
(a) Let $W\subset V$ be a subset. Show that W is a subspace of V if and only if the following two conditions hold:
(1) $\underline{0}\in W$
(2) For all $\underline{u},\underline{v}\in W$ and $a,b\in\mathbb{R}$ we have $a\underline{u}+b\underline{v}\in W$.
(b) Now let $W\subset V$ be a subspace. For any $n\geq 0$ let $\{\underline{w}_i\}_{i=1}^n\subset W$ be some vectors and let $\{a_i\}_{i=1}^n\subset\mathbb{R}$ be some scalars. Give an informal argument showing $\sum_{i=1}^n a_i\underline{w}_i=a_1\underline{w}_1+\cdots+a_n\underline{w}_n\in W$.
BONUS Give a formal proof by induction on n.
RMK The last item is intended as a diagnostic to see how many participants can write a proof by induction.

3. (A chain of subspaces)
   (a) Show that the space of bounded functions on a set X,
   $$\ell^\infty(X) \stackrel{\text{def}}{=} \left\{ f \in \mathbb{R}^X \mid \text{There is } M \in \mathbb{R} \text{ so that for all } x \in X, \text{ we have} |f(x)| \le M \right\},$$
   is a subspace of $\mathbb{R}^X$.
   (b) State (or reconstruct) theorems from calculus to the effect that "the space of convergent sequences, $c \stackrel{\text{def}}{=} \left\{ \underline{a} \in \mathbb{R}^{\mathbb{N}} \mid \lim_{n\to\infty} a_n \text{ exists} \right\}$, is a subspace of $\ell^\infty(\mathbb{N})$".
   RMK If you haven't seen those theorems before you can write them down first and then confirm their existence in your calculus textbook or Wikipedia. Don't forget that subspaces are subsets!
   (c) Show that the space of sequences of finite support, $\mathbb{R}^{\oplus\mathbb{N}} \stackrel{\text{def}}{=} \left\{ \underline{a} \in \mathbb{R}^{\mathbb{N}} \mid a_i \neq 0 \text{ for finitely many } i \right\}$, is a subspace of c. [now you need to know a little about convergent sequences]
   RMK Note that some vectors spaces have complicated names such as $\ell^\infty(X)$ or $\mathbb{R}^{\oplus\mathbb{N}}$. In either case you don't need to know why these were chosen – only what the space *is,* which is specified in the question.

**4. (§1.3 E19) Let V be a vector space and let W_1, W_2 be subspaces of V. Suppose that union $W_1 \cup W_2 = \{v \in V \mid v \in W_1 \text{ or } v \in W_2\}$ is a subspace of V (note that "or" includes the possibility that both assertions hold). Show that $W_1 \subset W_2$ or $W_2 \subset W_1$.

New spaces from old ones

Introduction: The previous problem and the following two are fairly difficult, at the higher level of skills for this course. We are practicing the skill of seeing definitions in class, and applying them by creating new objects and checking if they qualify. The problems are similar in spirit to problems 1,2 but harder. Problems 4 is challenging: it takes an *idea* to solve it. Problem 5 is *tedious*: you have to work through all the axioms and see if they hold. Together they help you practice *asbract algebra:* working with a vector space V without knowing what it is, purely based on axioms. One skill you will need is *unwinding definitions:* to write the arguments you will often need to replace the statement "V is a vector space" or "W is a subspace" with the properties that make V a vector space or W a subspace.

5. Let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be two vector spaces. On the set of pairs $V \times W = \{(\underline{v}, \underline{w}) \mid \underline{v} \in V, \underline{w} \in W\}$ define $(\underline{v}_1, \underline{w}_1) + (\underline{v}_2, \underline{w}_2) = (\underline{v}_1 +_V \underline{v}_2, \underline{w}_1 +_W \underline{w}_2)$ and $a \cdot (\underline{v}_1, \underline{w}_1) = (a \cdot_V \underline{v}_1, a \cdot_W \underline{w}_1)$. Show that this endows $V \times W$ with the structure of a vector space. We will call this space the *external direct sum* of V, W and denote it $V \oplus W$.

Challenge problems begin here

C1. Let W_1, W_2 be two subspaces of a vector space V.
   (a) Define their *internal sum* to be the set $W_1 + W_2 \stackrel{\text{def}}{=} \{\underline{w}_1 + \underline{w}_2 \mid \underline{w}_i \in W_i\}$. Show that $W_1 + W_2$ is a subspace of V.
   (*b) Show that $W_1 \cap W_2 = \{\underline{0}\}$ if and only if every vector in $W_1 + W_2$ has a *unique* representation in the form $\underline{w}_1 + \underline{w}_2$.

RMK In the case the equivalent conditions of (b) hold, we say that $W_1 + W_2$ is the *internal direct sum* of W_1, W_2 and confusingly also denote this space $W_1 \oplus W_2$. We will show later that in this case the two "direct sums" produced by problems 8 and 9(b) are in some sense the same. In general it will be possible to tell from context which direct sum is intended.

C2. Review supplementary problems C,D,E about fields. Let F be a field with finitely many elements. For an integer $n \geq 0$ write $\bar{n} = \sum_{i=1}^{n} 1_F$.
- (a) Show that $\bar{n} = \bar{m}$ for some $n > m > 0$ and conclude that $\bar{p} = 0_F$ for some positive integer p.
- (b) Show that the smallest positive p such that $\bar{p} = 0_F$ is a prime number. This is called the *characteristic* of F and denoted $\mathrm{char}(F)$.
- (*c) Show that $\{\bar{i} \mid 0 \leq i < \mathrm{char}(F)\}$ is a subfield of F, usually denoted the *prime field* of F.
- RMK We will later show that if F has characteristic p then its number of elements is of the form $q = p^f$ for some integer f.

DEFINITION. A vector space over the field F has the same definition as given in class, except that the field of scalars $\mathbb{R}$ is replaced with F.

C3*. Let E be a field, and let $F \subset E$ be a *subfield*. Show that the usual arithmetic operations of E endow it with the structure of a vector space over the field F.
- RMK This problem requires time to wrap your head around. It's the beginning of *field theory*: the study of fields, more deeply pursued in MATH 422.

Supplementary problems: abstractions

A. Write B^A for the set of all functions from the set A to the set B.
- (a) Let a' not be an element of A, and let $A' = A \cup \{a'\}$ be the set you get by adding a' to A. Construct a bijection between $B^{A'}$ and the set of pairs $B^A \times B = \left\{(f,b) \mid f \in B^A,\, b \in B\right\}$.
- (b) Suppose that A, B are finite sets. Show that $\#\left(B^A\right) = (\#B)^{(\#A)}$ where $\#X$ denotes the number of elements of a set X and on the right we have exponentiation of natural numbers.
  Hint: Induction on $\#A$.
- RMK Make sure to account for the corner cases where at least one the sets A, B is empty!

B. (Direct products and sums in general)
- (a) Let $\{V_i\}_{i\in I}$ be a family of vector spaces, and let $\prod_{i\in I} V_i$ (their *direct product*) denote the set $\{f \colon I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i\}$ (that is, the set of functions f with domain I such that $f(i)$ is an element of V_i for all i). For $f, g \in \prod_{i\in I} V_i$ and $a, b \in \mathbb{R}$ define $af + bg$ by $(af + bg)(i) = af(i) + bg(i)$ (addition and multiplication in V_i). Show that this endows $\prod_{i\in I} V_{i\in I}$ with the structure of a vector space.
- (b) Continuing with the same family, define the *support* of $f \in \prod_i V_i$ as $\mathrm{supp}(f) = \left\{i \in I \mid f(i) \neq \underline{0}_{V_i}\right\}$. Show that the *direct sum* $\bigoplus_{i\in I} V_i \stackrel{\text{def}}{=} \{f \in \prod_i V_i \mid \mathrm{supp}(f) \text{ is finite}\}$ is a subspace (compare with problem 6(c)).
- (c) When all the V_i are equal to a fixed space V we sometimes write V^I for the direct product $\prod_{i\in I} V$, and $V^{\oplus I}$ for the direct sum $\oplus_{i\in I} V$. Verify that this agrees with the notation in 6(c). What is V there?

Supplementary problems: fields

Notation: $\forall$ means "For all" and $\exists$ mean "there exists".

DEFINITION. A *field* is a triple $(F,+,\cdot)$ of a set F and two binary operations on F so that there are elements $0,1\in F$ for which:

$$\forall x,y,z\in F\ :\ x+y=y+x,\ (x+y)+z=x+(y+z),\ x+0=x,\ \exists x':x+x'=0$$
$$\forall x,y,z\in F\ :\ x\cdot y=y\cdot x,\ (x\cdot y)\cdot z=x\cdot(y\cdot z),\ x\cdot 1=x,\ (x\neq 0)\Rightarrow\exists\tilde{x}:x\cdot\tilde{x}=1$$
$$\forall x,y,z\in F: x\cdot(y+z)=x\cdot y+x\cdot z$$

In other words, all the usual laws of arithmetic hold *including* the fact that we can divide by non-zero numers.

EXAMPLE. The field of real numbers $\mathbb{R}$, the field of rational numbers $\mathbb{Q}$, the field of complex numbers $\mathbb{C}$. The field of rational functions $\mathbb{R}(x)=\left\{\frac{f}{g}\mid f,g \text{ real polynomials with } g\neq 0\right\}$.

C. (Elementary calculations) Let F be a field.
  (a) Let $0_1,0_2$ be two elements of F which can be used in the definition above. By considering the sum 0_1+0_2 show that $0_1=0_2$.
  (b) Let $x\in F$ and let $x_1',x_2'\in F$ be such that $x+x_1'=x+x_2'=0$. Adding x_1' to both sides conclude that $x_1'=x_2'$. This element is usually denoted $-x$.
  (c) Let $x\in F$. Show that $0\cdot x=0$.
  (d) Similarly show that 1 and $\tilde{x}$ (usually denoted x^{-1}) are unique.
  (e) Show that if $xy=0$ then $x=0$ or $y=0$.

D. Consider the set $\{0,1\}$ with $0\neq 1$. Define $1+1=0$, and define all other sums and products in this set as required by the definition above or by C(c). Show that the result is a field. Show that defining $1+1=1$ would not result in a field, and conclude that there is a unique field with two elements, denoted $\mathbb{F}_2$ from now on.

E. Let X be a set. To a subset $A\subset X$ associate its *indicator function* $1_A(x)=\begin{cases}1 & x\in A\\ 0 & x\notin A\end{cases}$. Show that the map $A\mapsto 1_A$ gives a bijection between the *powerset* $\mathcal{P}(X)=\{A\mid A\subset X\}$ and the vector space $\mathbb{F}_2^X$. Show that under this identification addition in $\mathbb{F}_2^X$ maps to the operation *of symmetric difference* of sets, defined by $A\Delta B=\{x\mid x\in A\cup B,\ x\notin A\cap B\}$ (that is, $A\Delta B$ is the set of elements of X that are in *exactly one* of A,B but not both).