Lior Silberman's Math 223: Problem Set 4 (due 9/2/2022)

Practice problems (recommended, but do not submit)

Section 2.1, Problems 1-3,5,9,10-12,28-29
Section 2.2, Problems 1-3.

Calculations with linear maps

M1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_1-x_2 \\ 2x_1 \end{pmatrix}$.

(a) Find bases for $\operatorname{Ker} T$, $\operatorname{Im} T$ and check that the dimension formula holds.

(b) Find the matrix for T with respect to the bases $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

M2. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_1+x_2 \\ x_1-x_2+x_3-x_5 \\ -3x_1-x_3+x_5 \end{pmatrix}$.

(a) Find bases for $\operatorname{Ker} T$, $\operatorname{Im} T$ (use problem 1) and check that the dimension formula holds.
(b) Find the matrix for T with respect to the standard bases of $\mathbb{R}^5$, $\mathbb{R}^3$.
(c) Find the matrix for T with respect to the standard basis of $\mathbb{R}^5$ and the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

More linear maps

1. Let $D\colon \mathbb{R}[x]^{\leq n} \to \mathbb{R}[x]^{\leq n}$ be the differentiation map.
(a) Find $\operatorname{Ker} D$ and its dimension.
(b) Find $\operatorname{Im} D$.
Fix a number $a \neq 0$ and let $T\colon \mathbb{R}[x]^{\leq n} \to \mathbb{R}[x]^{\leq n}$ be the map $D + Z_a$ (that is, $Tp = \frac{dp}{dx} + a \cdot p$ for any polynomial p).
(c) Show that T maps the basis of monomials to a set of $n+1$ polynomials of distinct degrees.
(*d) Show that $\operatorname{Im} T = \mathbb{R}[x]^{\leq n}$.

2. Let V be a vector space. For $A, B \in \operatorname{Hom}(V,V)$ define $A \cdot B = A \circ B$; in class we checked that $A \cdot B \in \operatorname{Hom}(V,V)$.
RMK In PS3 Problem C2 we checked that $\operatorname{Hom}(V,V)$ is a vector space under pointwise addition.
(a) Check that the multiplication we define is associative: $(AB)C = A(BC)$ (hint: evaluate both sides at $\underline{v} \in V$), and that the identity map Id_V is a unit for it.
(b) Check that this multiplication is distributive over addition: $(A+B)C = AC + BC$, $C(A+B) = CA + BC$.
DEF For any two linear maps $A, B \in \operatorname{Hom}(V,V)$ their *commutator* is the linear map $[A,B] = AB - BA$.
(c) Show that $A \cdot B = B \cdot A$ iff $[A,B] = 0$ (hence the name "commutator").
PRAC For a *function* $a \in C^\infty(\mathbb{R})$ write M_a for the operator of *multiplication by a:* $(M_a f)(x) = a(x) f(x)$. Show that $M_a\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a linear map.
(d) Let $a \in C^\infty(\mathbb{R})$. Find a function $b \in C^\infty(\mathbb{R})$ so that $[D, M_a] = M_b$ as linear maps on $C^\infty(\mathbb{R})$.

Surjective and injective maps; Invertibility

DEFINITION. Let $T\colon U \to V$ be a linear map. We say that T is *injective* (a *monomorphism*) if $T\underline{u} = T\underline{u}'$ implies $\underline{u} = \underline{u}'$ and *surjective* (an *epimorphism)* if $\operatorname{Im} T = V$. If a linear map $T\colon U \to V$ is surjective and injective we say it is an *isomorphism* (of vector spaces). We say that U, V are *isomorphic* if there is an isomorphism between them.

3. Show that T is injective if and only if $\operatorname{Ker} T = \{\underline{0}\}$. (Hint: to compare two vectors consider their difference)

4. Suppose that $T\colon U \to V$ is an isomorphism of vector spaces, and define a function $T^{-1}\colon V \to U$ as follows: $T^{-1}\underline{v}$ is that vector $\underline{u}$ such that $T\underline{u} = \underline{v}$.
   (a) Explain why $\underline{u}$ exists and why it is unique (that is, check that T^{-1} is a well-defined function).
   (*b) Show that T^{-1} is a linear function.

Extra credit: categorical thinking

C1. Let $T \in \operatorname{Hom}(U,V)$, $S \in \operatorname{Hom}(V,U)$ be linear maps.
   (a) Suppose $TS = \operatorname{Id}_V$. Show that S is injective and T is surjective.
   (b) (Converse of 4(b)) Suppose that $TS = \operatorname{Id}_V$ and $ST = \operatorname{Id}_U$. Show that T is an isomorphism and that $S = T^{-1}$.

C2. Let $T \in \operatorname{Hom}(U,V)$.
   (a) Show that T is injective if and only if for every vector space Z and every two linear maps $f_1, f_2\colon Z \to U$ if $T \circ f_1 = T \circ f_2$ then $f_1 = f_2$.
   (**b) Show that T is surjective if and only if for every vector space Z and every two linear maps $f_1, f_2\colon V \to Z$ if $f_1 \circ T = f_2 \circ T$ then $f_1 = f_2$.