Lior Silberman’s Math 223: Problem Set 4 (due 9/2/2022)

Calculations with linear maps

M1. Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be the linear map \( T \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 \end{array} \right) \).

(a) Find bases for \( \text{Ker} T, \text{Im} T \) and check that the dimension formula holds.

(b) Find the matrix for \( T \) with respect to the bases \( \{ \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \} \) of \( \mathbb{R}^2 \) and \( \{ \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \} \)

of \( \mathbb{R}^3 \).

M2. Let \( T : \mathbb{R}^5 \to \mathbb{R}^3 \) be the linear map \( T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right) = \left( \begin{array}{c} 2x_1 + x_2 \\ x_1 - x_2 + x_3 - x_5 \\ -3x_1 - x_3 + x_5 \end{array} \right) \).

(a) Find bases for \( \text{Ker} T, \text{Im} T \) (use problem 1) and check that the dimension formula holds.

(b) Find the matrix for \( T \) with respect to the standard bases of \( \mathbb{R}^5, \mathbb{R}^3 \).

(c) Find the matrix for \( T \) with respect to the standard basis of \( \mathbb{R}^5 \) and the basis \( \{ \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right) \} \)

of \( \mathbb{R}^3 \).

More linear maps

1. Let \( D : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n} \) be the differentiation map.
   (a) Find \( \text{Ker} D \) and its dimension.
   (b) Find \( \text{Im} D \).
   Fix a number \( a \neq 0 \) and let \( T : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n} \) be the map \( D + Z_a \) (that is, \( Tp = \frac{dp}{dx} + a \cdot p \) for any polynomial \( p \)).
   (c) Show that \( T \) maps the basis of monomials to a set of \( n + 1 \) polynomials of distinct degrees.
   (*d) Show that \( \text{Im} T = \mathbb{R}[x]_{\leq n} \).

2. Let \( V \) be a vector space. For \( A, B \in \text{Hom}(V, V) \) define \( A \cdot B = A \circ B \); in class we checked that \( A \cdot B \in \text{Hom}(V, V) \).
   RMK In PS3 Problem C2 we checked that \( \text{Hom}(V, V) \) is a vector space under pointwise addition.
   (a) Check that the multiplication we define is associative: \( (AB)C = A(BC) \) (hint: evaluate both sides at \( v \in V \), and that the identity map \( \text{Id}_V \) is a unit for it.
   (b) Check that this multiplication is distributive over addition: \( (A + B)C = AC + BC, C(A + B) = CA + BC \).
   DEF For any two linear maps \( A, B \in \text{Hom}(V, V) \) their commutator is the linear map \( [A, B] = AB - BA \).
   (c) Show that \( A \cdot B = B \cdot A \) iff \( [A, B] = 0 \) (hence the name “commutator”).
   PRAC For a function \( a \in C^{\infty}(\mathbb{R}) \) write \( M_a \) for the operator of multiplication by \( a \): \( (M_a f)(x) = a(x)f(x) \). Show that \( M_a : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R}) \) is a linear map.
   (d) Let \( a \in C^{\infty}(\mathbb{R}) \). Find a function \( b \in C^{\infty}(\mathbb{R}) \) so that \([D, M_a] = M_b\) as linear maps on \( C^{\infty}(\mathbb{R}) \).
Surjective and injective maps; Invertibility

**Definition.** Let $T : U \to V$ be a linear map. We say that $T$ is injective (a monomorphism) if $Tu = Tu'$ implies $u = u'$ and surjective (an epimorphism) if $\text{Im} T = V$. If a linear map $T : U \to V$ is surjective and injective we say it is an isomorphism (of vector spaces). We say that $U, V$ are isomorphic if there is an isomorphism between them.

3. Show that $T$ is injective if and only if $\text{Ker} T = \{0\}$. (Hint: to compare two vectors consider their difference)

4. Suppose that $T : U \to V$ is an isomorphism of vector spaces, and define a function $T^{-1} : V \to U$ as follows: $T^{-1}v$ is that vector $u$ such that $Tu = v$.
   (a) Explain why $u$ exists and why it is unique (that is, check that $T^{-1}$ is a well-defined function).
   (*b) Show that $T^{-1}$ is a linear function.

**Extra credit: categorical thinking**

C1. Let $T \in \text{Hom}(U, V), S \in \text{Hom}(V, U)$ be linear maps.
   (a) Suppose $TS = \text{Id}_V$. Show that $S$ is injective and $T$ is surjective.
   (b) (Converse of 4(b)) Suppose that $TS = \text{Id}_V$ and $ST = \text{Id}_U$. Show that $T$ is an isomorphism and that $S = T^{-1}$.

C2. Let $T \in \text{Hom}(U, V)$.
   (a) Show that $T$ is injective if and only if for every vector space $Z$ and every two linear maps $f_1, f_2 : Z \to U$ if $T \circ f_1 = T \circ f_2$ then $f_1 = f_2$.
   (**b) Show that $T$ is surjective if and only if for every vector space $Z$ and every two linear maps $f_1, f_2 : V \to Z$ if $f_1 \circ T = f_2 \circ T$ then $f_1 = f_2$. 