Lior Silberman's Math 223: Problem Set 5 (due 16/2/2022)

Calculations with matrices

M1. Let $A=\begin{pmatrix}-2 & 3\\ 5 & -7\end{pmatrix}$, $B=\begin{pmatrix}4 & 1 & 0\\ 0 & -2 & 9\end{pmatrix}$, $C=\begin{pmatrix}0 & 0\\ 1 & 1\\ 2 & 2\end{pmatrix}$, $D=\begin{pmatrix}7 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 5\end{pmatrix}$. Calculate all possible products among pairs of A,B,C,D (don't forget that $A^2=AA$ is also such a product and that XY,YX are different products if both make sense).

M2. The $n\times n$ *identity matrix* is the matrix $I_n\in M_n(\mathbb{R})$ with entries: $(I_n)_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$. Show that $I_n\underline{v}=\underline{v}$ for all $\underline{v}\in\mathbb{R}^n$.

M3. Let $A\in M_{m,n}(\mathbb{R})$. Show that $AI_n=I_mA=A$.

M4. (a) Let $A\in M_{n,m}(\mathbb{R})$, $B\in M_{m,p}(\mathbb{R})$. Show that the jth column of AB is given by the product $A\underline{v}$ where $\underline{v}$ is the jth column of B.
(b) Let $A\in M_{n,m}(\mathbb{R})$, $B\in M_{m,p}(\mathbb{R})$. Show that the jth column of AB is a linear combination of all the columns of A with the coefficients being the jth column of B.

M5. ("Group homomorphisms")
(a) Let R_α be the matrix $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ ("rotation in the plane by angle α"). Show that $R_\alpha R_\beta=R_{\alpha+\beta}$.
(b) Let $n(x)$ be the matrix $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ ("shear in the plane by x"). Show that $n(x)n(y)=n(x+y)$.

More calculations

1. Let $A,B\in M_n(\mathbb{R})$ be square matrices. We say A,B *commute* if $AB=BA$. We say A is *scalar* if $A=zI_n$ for some $z\in\mathbb{R}$. The *centre* of $M_n(\mathbb{R})$ is the set $Z=\{A\in M_n(\mathbb{R})\mid \forall B\in M_n(\mathbb{R}):AB=BA\}$ of matrices that commute with all other matrices.
PRAC Check that the action of zI_n on vectors is by multiplication by the scalar z.
(a) Show that $Z\subset M_n(\mathbb{R})$ is a subspace.
(b) Show that the centre of $M_n(\mathbb{R})$ consists of scalar matrices: $Z=\mathrm{Span}_{\mathbb{R}}(I_n)$.

2. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R})$ and suppose that $ad-bc\neq 0$.
(a) Find a matrix $B=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$ such that $AB=I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (express the entries e,f,g,h in terms of a,b,c,d). Show that $BA=I_2$ as well.
(*b) ("Uniqueness of the inverse") Suppose that $AC=I_2$. Show that $C=B$.

Composition of linear maps

3. $B\in\mathrm{Hom}(U,V)$ and $A\in\mathrm{Hom}(V,W)$ be linear maps (so that the composition $AB\in\mathrm{Hom}(U,W)$ makes sense).
(a) Show that $\mathrm{Ker}(AB)\supset\mathrm{Ker}(B)$ and that $\mathrm{Im}(AB)\subset\mathrm{Im}(A)$. Conclude that if AB is injective then so is B and if AB is surjective then so is A.
(b) Suppose A,B are both injective or both surjective. In each case show that AB has the same property.

4. (Nilpotency) Fix a vector space V. A linear map $T \in \mathrm{End}(V)$ is called *nilpotent* if $T^k = 0$ for some k. The smallest such k is called the *degree of nilpotency* of T.
   (*a) Find a matrix $N \in M_2(\mathbb{R})$ such that $N^2 = 0$ but $N \neq 0$.
   (**b) Let $T \in \mathrm{End}(V)$ be nilpotent, and suppose that $\dim V = n < \infty$. Show that $T^n = 0$.
   Hint: consider the kernels of successive powers of T.

Extra credit

C1. (Projections) Call $P \in \mathrm{End}(V)$ a *projection* if $P^2 = P$. Fix a projection P.
   (a) Let $U = \mathrm{Im}(P) \subset V$. Show that $P\underline{u} = \underline{u}$ for all $\underline{u} \in U$.
   (*b) Let $U^\perp = \mathrm{Ker}(P)$. Show that $V = U + U^\perp$ and that $U \cap U^\perp = 0$.
   RMK1 We aren't implying that $U^\perp$ is "orthogonal" to U here – there is no notion of angle in our vector space!
   RMK2 In PS1 problem C1(b) we showed that the conclusion of part (b) implies that every vector $\underline{v} \in V$ has a *unique* representation in the form $\underline{v} = \underline{u} + \underline{u}^\perp$ where $\underline{u} \in U$ and $\underline{u}^\perp \in \underline{u}^\perp$. Then
$$P\underline{v} = P\left(\underline{u} + \underline{u}^\perp\right) = P\underline{u} + P\underline{u}^\perp = \underline{u} + \underline{0} = \underline{u}.$$
   (c) Conversely let $W, W^\perp \subset V$ be any two subspaces such that $V = W + W^\perp$ and $W \cap W^\perp = 0$. Show that there is a unique linear map $P\colon V \to V$ such that $P\restriction_W = \mathrm{Id}_W$ and $P\restriction_{W^\perp} = 0_{W^\perp}$, and that P is a projection (hint: check that P^2 satisfies the same conditions)
   DEF We say that P is the projection *onto U along $U^\perp$* $= \mathrm{Ker}(P)$.
   (d) Let $B \subset U$ and $B^\perp \subset U^\perp$ be bases, so that $B \cup B^\perp$ is a basis of V (try proving that as an extra exercise). Show that the matrix of P in this basis is block diagonal: in the block corresponding to B we have an identity matrix and elsewhere we have zeroes.
   (e) Show that $P^\perp = \mathrm{Id}_V - P$ is also a projection (called the *complementary projection*).
   (f) Show that $P^\perp$ is the projection onto $U^\perp$ along U.

Supplementary problem

A. Show by hand that for any three matrices A, B, C with compatible dimensions $(AB)C = A(BC)$.

Supplementary problems: the formalism of infinite sums

We introduce notation to make sense of infinite linear combinations of vectors, as long as only finitely many summands are nonzero. This will unify our notations for finite-dimensional and infinite-dimensional spaces. *Warning*: the claims often appear obvious, but this is an artefact of our choice of notation; the *proofs* justify the notation by showing that it behaves as expected.

Fix a vector space V.

B. (a) Let I be an index set. Define the *support* of $f \in V^I$ to be $\mathrm{supp} f = \{i \in I \mid f(i) \neq \underline{0}\}$. Show that $V^{\oplus I} \stackrel{\text{def}}{=} \left\{f \in V^I \mid \mathrm{supp} f \text{ is finite}\right\}$ is a subset of V^I.
   (b) Define a map
$$\sum_{i \in I}\colon V^{\oplus I} \to I$$
   given by
$$\sum_{i \in I} f = \sum_{i \in \mathrm{supp} f} f(i).$$
   Show that in fact $\sum_{i \in I} f = \sum_{i \in J} f$ for any finite subset $J \subset I$ containing the support of f.
   (c) Show that $\sum_{i \in I}$ is a linear map. We will generally write $\sum_{i \in I} f(i)$ rather than $\sum_{i \in I} f$ for this map.

RMK This gives us two desirable properties of sums: $\sum_{i\in I} f(i) + \sum_{i\in I} g(i) = \sum_{i\in I}(f(i)+g(i))$ and $a\sum_{i\in I} f(i) = \sum_{i\in I} a\cdot f(i)$.

(d) Let I,J be disjoint, and let $f\in V^{\oplus I}$, $g\in V^{\oplus J}$. Let $h\colon I\cup J\to V$ be the union $f\cup g$, that is the function

$$h(i) = \begin{cases} f(i) & i\in I \\ g(i) & i\in J \end{cases}.$$

Show that h has finite support and that

$$\sum_{i\in I\cup J} h(i) = \left(\sum_{i\in I} f(i)\right) + \left(\sum_{j\in J} g(j)\right).$$

RMK This is the other desirable property of sums: *concatenation*.

C. (Review of linear algebra) Fix a vector space V. We show that with the formalizm of problem B, we can ignore a lot of the distinction between finite-dimensional and infinite-dimensional spaces.

(a) Let $\{\underline{v}_i\}_{i\in I}\subset V$ be a parametrized set of vectors (i.e. a function $I\to V$). Show that

$$\mathrm{Span}_F\left(\{\underline{v}_i\}_{i\in I}\right) = \left\{\sum_{i\in I} a_i\underline{v}_i \mid a_i\in\mathbb{R}^{\oplus I}\right\}.$$

Note that this removes the need to choose finitely many vectors in $\underline{v}_i$.

(b) Call $\{\underline{v}_i\}_{i\in I}$ *linearly dependent* if there is $j\in I$ such that $\underline{v}_j\in\mathrm{Span}_F\left(\{\underline{v}_i\}_{i\in I\setminus\{j\}}\right)$. Show that if $\underline{v}_i=\underline{v}_j$ for some $i\neq j$ then the family is linearly dependent.

RMK Unlike a set of vectors, a parametrized set really is a function, so we can have repetitions.

(c) Show that $\{\underline{v}_i\}_{i\in I}$ is independent iff for all $\underline{a}\in\mathbb{R}^{\oplus I}$ if $\sum_i a_i\underline{v}_i=\underline{0}$ then all $a_i=0$.

(d) Suppose $\{\underline{v}_i\}_{i\in I}\subset V$ is independent and spanning in the above sense. Show that the $\underline{v}_i$ are distinct, and that the resulting *set* (the image of the function) is a basis in the ordinary sense.

DEF We call such a parametrized set an *ordered basis* or a *parametrized basis*.

RMK Conversely every set can be made into a parametrized set by using the set itself as the index set (and using the identity function).

D. (Every vector space is $\mathbb{R}^n$) Let V be a vector space with basis $B=\{\underline{v}_i\}_{i\in I}$ (I may be infinite).

(a) Let $\Phi\colon\mathbb{R}^{\oplus I}\to V$ be the map $\Phi(\underline{a})=\sum_{i\in I} a_i\underline{v}_i$. Show that Φ is a an isomorphism of vector spaces (part of this was problem C(a)).

RMK The inverse map $\Psi\colon V\to\mathbb{R}^{\oplus I}$ is called the *coordinate map* (in the ordered basis B)

(b) Construct an isomorphism $V^*\to\mathbb{R}^I$.

(c) Let W be another space with ordered basis $C=\left\{\underline{w}_j\right\}_{j\in J}$. Construct an injective linear map $\mathrm{Hom}(V,W)\to M_{I\times J}(\mathbb{R})=\mathbb{R}^{I\times J}$ and show that its image is the set of matrices having at most finitely many non-zero entries in each column.