Lior Silberman's Math 223: Problem Set 6 (due 28/2/2022)

Practice problems (recommended, but do not submit)

P1. Let U,V,W,X be vector spaces.
  (a) Let $A \in \mathrm{Hom}(U,V)$, $B \in \mathrm{Hom}(W,X)$. We define maps $R_A\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(U,W)$, $L_B\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,X)$ and $S_{A,B}\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(U,X)$ by $R_A(T) = TA$, $L_B(T) = BT$, $S_{A,B}(T) = BTA$. Show that all three maps are linear.
  (b) Suppose that $A,B \in \mathrm{Hom}(U,U)$ are invertible, with inverses A^{-1}, B^{-1}. Show that AB is invertible, with inverse $B^{-1}A^{-1}$ (note the different order!)

P2. Let U,V be vector spaces and let $B \subset U$ be a basis.
  (*a) Let $f \in \mathrm{Hom}(U,V)$ be a linear isomorphism. Show that the image $f(B) = \{f(\underline{v}) \mid \underline{v} \in B\}$ is a basis of V.
  RMK It follows that is U is isomorphic to V then $\dim U = \dim V$.
  (**b) Conversely, suppose that $B' \subset V$ is a basis, and and that $g\colon B \to B'$ is a function which is $1-1$ and onto (see notations file). Show that there is an isomorphism $f \in \mathrm{Hom}(U,V)$ which agrees with g on B.
  RMK It follows that if $\dim U = \dim V$ then U is isomorphic to V.

M1. Create several systems of linear equations in several unknowns and solve them using Gaussian elimination.

Linear equations

1. (Recognition) Express the following equations as linear equations by finding appropriate spaces, linear map, and constant vector.
  (a) $\begin{cases} 5x+7y & = 3 \\ z+2x & = 1 \\ 2y+x+3z & = -1 \\ x+y & = 0 \end{cases}.$
  (b) (Bessel equation) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2-\alpha^2)y = 0$. Use the space $C^\infty(\mathbb{R})$ of functions on $\mathbb{R}$ which can be differentiated to all orders.
  (*c) Fixing $S,B \in \mathrm{Hom}(U,U)$ with S invertible, $SXS^{-1} = B$ for an unknown $X \in \mathrm{Hom}(U,U)$ (Show that the map you define is linear!)
  PRAC Suppose that $\dim U = n$. Using a basis for U, replace the equation of (c) with a system of n^2 equations in n^2unknowns.

2. (Solution) Write the extended matrix for the system of equations $\begin{cases} 5x+3y+2z & = 3 \\ 2x+y+5z & = 1 \\ y+2z+w & = 2 \end{cases}$ and then solve the system using Gaussian elimination.

Similarity of matrices

3. Suppose that $\dim U = \dim V < \infty$. Let $A \in \mathrm{Hom}(U,V)$. Show that the following are equivalent:
  (1) A is invertible.
  (2) A is surjective.
  (3) A is injective.
  Hint: use the rank-nullity theorem.

Recall the notation $\mathrm{End}(U)$ for $\mathrm{Hom}(U,U)$ (linear maps from U to itself)..

DEFINITION. We say that two transformations $A,B \in \mathrm{End}(U)$ are *similar* if there is an invertible transformation $S \in \mathrm{End}(U)$ such that $B = SAS^{-1}$.

4. (Calculations)
   PRAC Suppose that A,B are similar and $A = 0$. Show that $B = 0$.
   (a) Suppose that A,B are similar and $A = \mathrm{Id}_U$. Show that $B = \mathrm{Id}_U$.
   (b) Show that the matrices $A = \begin{pmatrix} 0 & 2 \\ 6 & -4 \end{pmatrix}$, $B = \begin{pmatrix} -33 & 15 \\ -63 & 29 \end{pmatrix}$ are similar via the similarity transformation $S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. (For a formula for S^{-1} see PS5)

For the rest of the problem set fix A,B,S such that $B = SAS^{-1}$.

5. (One meaning of similarity) Let $\mathcal{B} = \{\underline{v}_i\}_{i\in I} \subset U$ be a basis. By a practice problem P2 above, $\mathcal{B}' = \{S\underline{v}_i\}_{i\in I} \subset U$ is also a basis. Let $M \in M_I(\mathbb{R})$ be the matrix of A with respet to the basis $\mathcal{B}$. Show that M is also the matrix of $B = SAS^{-1}$ with respect to the basis $\mathcal{B}'$.
   RMK We'll show later that similarity of *matrices* has another, different meaning: similar matrices represent *the same* transformation with respect to *different* bases.

6. (Similarity is an "equivalence relation")
   (a) show that A is similar to A for all A. (Hint: choose S wisely)
   (b) Suppose that A is similar to B. Show that B is similar to A (Hint: solve $B = SAS^{-1}$ for A).
   (c) Suppose that A is similar to B, and B is simlar to C. Show that A is similar to C.

Extra credit: Polynomials and matrices

DEFINITION. For $A \in \mathrm{End}(A)$ set $A^0 = \mathrm{Id}_U$ and then define A^n recursively via $A^{n+1} = A^n \cdot A$. For a polynomial $p(x) = \sum_{i=0}^n a_i x^i \in \mathbb{R}[x]$ define $p(A) = \sum_{i=0}^n a_i A^i$.

C1. Let $q(x) = \sum_{j=0}^m b_j x^j \in \mathbb{R}[x]$ be another polynomial, and let $s(x) = p(x) + q(x)$, $r(x) = p(x)q(x)$ be their sum and product in $\mathbb{R}[x]$. Show that $s(A) = p(A) + q(A)$ and that $r(A) = p(A)q(A)$.
   RMK1 This problem seems silly, but checking that things work "the way they are supposed to" is important. To understand the motivation note that we think of polynomial as *formal expressions* rather than functions – we need to make a *definition* to interpret them as functions, and then we need to verify that this definition works as expected.
   RMK2 We have proven that the map $\mathbb{R}[x] \to \mathrm{End}(U)$ given by $p \mapsto p(A)$ is a "homomorphism of $\mathbb{R}$-algebras".

C2. (Induction practice)
   (a) Show that $B^n = SA^nS^{-1}$ for all n.
   (b) Show that $p(B) = Sp(A)S^{-1}$.